A STUDY ON OVERCONVERGENCE OF CERTAIN SEQUENCES OF POLYNOMIAL INTERPOLANTS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
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to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

JULY, 1995

Certificate

It is certified that the work contained in this thesis entitled "**A study on Overconvergence of certain sequences of Polynomial Interpolants** ", by **Pratibha Dua** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree or diploma.

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July, 1995

ABSTRACT

Theorem of J.L.Walsh, involving the phenomenon of overconvergence, has develope a new branch in Approximation Theory. It was first extended by Cavaretta et al whic became the subject of extension of Walsh's theorem in various other directions In thi thesis an attempt has been made to generalise and extend few of them, further. Wals overconvergence by average of interpolating polynomials and their derivatives has bee studied in detail. Least square approximating polynomialds are considered seperatly an together with average of interpolating polynomials as well. In both the cases it has bee shown that results can be obtained for the differences of two sequences when the n^{th} roots c unity are replaced by n^{th} roots of α^n. Quantitative estimates are obtained for the deriva tive of the sequence of differences of polynomials obtained from Hermite interpolatio Equiconvergence results in the case of polynomial interpolants in z, z^{-1} has been extende and generalised for the sequence of polynomials in z and z^{-1} seperatly. Lastly functio analytic in an ellipse and hence represented by Chebyshev series are studied. Here th average of polynomials associated with zeros and extremas of Chebyshev polynomials a considered.

To

My Mother -

my source of inspiration.

ACKNOWLEDGEMENTS

I hereby take the apportunity to express my sincere gratitude and thanks to my thesis supervisor, Prof. O.P.Juneja for his valuable guidence during the period of present work. I owe to him much more than I can possibly express. His affectinate teachings are most possessed treasures of my life.

I greatfully acknowledge the help, enthusiasm and critical comments receieved from Drs. Manjul Gupta, Prabha Sharma, R.K.S.Rathore, Piyush Chandra, S.K.Gupta, Santokh Singh. I sincerely thanks all the faculty members and the non-teaching staff of Mathematics department for their help.

I can never say in words how greatful I am to my father, brother, sister, jija-ji,Ami and Abhilasha who have been a constant source of inspiration and encouragement throughout the course of this work. My special thanks are also due to my neighbours specially Mrs. A.B.L.Agrawal and Mrs.H.D.Bist, for the concern they showed for the progress of my work and always encouraged me during my difficult momemts. I appreciate the company of Aarti, Abha di, Abhishek, Ajay bhaiya, Anil, Apul, Ashu, Atul bhaiya, Chanduka, Himanshu, Kalika, Nisha, Rachna, Ratna, Piyush, Rangan, Vivek and Umesh bhaiya whose company provided a healthy and pleasant atmosphere for carrying out research and boosted my moral to my work.

I sincerely aknowledge the cooperation and silent encouragment given to me by my inlaws, Shummi and Nimmi. I would like to convey my deepest and most sincere gratitude for successfull completion of my thesis to my mother, Mrs Krishna Tulsidass, who took the burden off all my hardships, disappointments and sufferings upon herself and lived them with me - encouraging me to look beyond the troubles even in the darkest of the hour! Thank You Mom for always being there.

I record my sincere thanks to my husband, for his unfailing affection and care. The pateince shown by him is creditable and without whose perennial support it would not have been possible for me to complete this investigation.

The credit of thesis typing goes to a large extent to Dr. Shonmugraj, Joydeep and Rangan Guha, who introduced me to the world of LATEX. I am also thankful to the authorities of I.I.T. Kanpur for the facilities and the research fellowship of the institute. Thanks to all others who helped me in one way or other to carry out the present investigation.

Pratibha

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CONTENTS

SYNOPSIS

of the

Ph.D. Dissertation

on

A STUDY ON OVERCONVERGENCE OF CERTAIN SEQUENCES OF POLYNOMIAL INTERPOLANTS

submitted by

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to the

Department of Mathematics

Indian Institute of Technology, Kanpur

under the supervision of

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July, 1995

A power series is said to be overconvergent in a region G containing the circle of convergence of the series, if it is possible to select from the sequence of its partial sums a subsequence which converges almost uniformly in G.

In early thirties J.L.Walsh showed that the sequence of differences of the Lagrange interpolants in the roots of unity and the partial sums (of the same degree) of a function $f \in A_\rho$ (analytic in $|z| < \rho, \rho > 1$ but not in $|z| \leq \rho$) converges to zero in $|z| < \rho^2$. The essential feature of Walsh's theorem is that equiconvergence holds in the larger disk $|z| < \rho^2$. This result attracted little attention until early eighties, when a variety of its variations began to appear. In 1982 A.S.Cavaretta, A.Sharma and R.S.Varga generalized Walsh's result by comparing the interpolating polynomial of f to the polynomials determined from its power series expansion. In 1985 it was remarked by E.B.Saff and R.S.Varga that a counterpoint to the above generalization says that the sequence of differences in question can be bounded in at most one point of $|z| > \rho^2$ and, moreover, given a point in $\rho^2 < |z| < \rho^3$, there is an admissible f whose differences at that point tends to zero. In 1986 these estimates concerning the overconvergence of complex interpolating polynomials where further generalized by V.Totik and K.G.Ivanov & A.Sharma. Converse of the extension given by Cavaretta et al was proved by J.Szabados. T.E.Price considered the

average of interpolating polynomials to extend the theorem of Walsh. In 1990 Lou Yaunren investigated the pointwise estimates of the r^{th} derivative of the sequence of differences of two polynomials ($\Delta_{l,n-1}(z;f)$) associated with f. For any $f \in A_\rho$ and any positive integer l, the quantity $\Delta_{l,n-1}(z;f)$ was studied extensively by considering the least square approximation, Hermite interpolation, Hermite Birkhoff interpolation, polynomial interpolation in z, z^{-1} etc. In 1986 Lou Yuanren gave a new direction in Walsh equiconvergence theory by considering n^{th} roots of α^n, $|\alpha| < \rho$ which was further extended by several authors. In 1983 E.B.Saff and A.Sharma extended the walsh theory of overconvergence to rational interpolants. While dealing with rational interpolants, meromorphic functions were also considered by some authors. T.J.Rivlin considered functions analytic in an ellipse and obtained some preliminary results in this direction.

In this dissertation our effort has been to further investiagate the results for different sequences of polynomials in the direction of V.Totik, K.G.Ivanov & A.Sharma and Lou Yuanren by using the tools of Walsh overconvergence theory. The material of the thesis has been divided into seven chapters, a brief scenario of which we present now.

Chapter one sketches in brief the development of the work carried out in Walsh equiconvergence theory along with an outline of the contents of all chapters of the thesis.

In Chapter two an attempt is made to see how far results are valid for average of certain polynomials associated with a function in A_ρ. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $L_{n-1}(z;f)$ be the Lagrange interpolating polynomial of degree $(n-1)$ to the function f at n^{th} roots of unity. For m and n positive integers let $\omega = exp(2\pi i/mn)$. Set $f_q(z) = f(z\omega^q), q = 0, 1, \ldots, m-1$, and define the averages

$$A_{n-1,m}(z;f) = \frac{1}{m}\sum_{q=0}^{m-1} L_{n-1}(z\omega^{-q}; f_q)$$

and

$$A_{n-1,m,j}(z;f) = \frac{1}{m}\sum_{q=0}^{m-1} P_{n-1,j}(z\omega^{-q}; f_q), \qquad j = 0, 1, \ldots,$$

where $P_{n-1,j}(z;f) = \sum_{k=0}^{n-1} a_{k+nj} z^k$. Further, for any positive integer l denote

$$\Delta_{n-1,m,l}(z;f) = A_{n-1,m}(z;f) - \sum_{j=0}^{l-1} A_{n-1,m,j}(z;f).$$

Here we study the pointwise behaviour of the sequence $\{\Delta_{n-1,m,l}(z;f)\}$ and obtain some quantitative estimates of the quantity $\overline{\lim}_{n\to\infty}|\Delta_{n-1,m,l}(z;f)|^{1/n}$. We also consider the sequence $\{\Delta^{(r)}_{n-1,m,l}(z;f)\}$ which is the r^{th} derivative of $\{\Delta_{n-1,m,l}(z;f)\}$ and give some quan-

titative estimates of $\overline{\lim}_{n\to\infty} \max_{|z|=R} |\Delta^{(r)}_{n-1,m,l}(z;f)|^{1/n}$ and investigate some pointwise estimate of $\Delta^{(r)}_{n-1,m,l}(z;f)$. The concept of distinguished point of degree r and distinguished sets is also considered. For a special case, results for the derivatives (mentioned above) reproduce and generalise the few earlier results of the same chapter. Results of this chapter extend the results of T.E.Price [*j. Approx. Theory* 43 (1985), No.2, 140-150], V.Totik [*J. Approx. Theory* 47 (1986), No.3, 173-183], Ivanov & Sharma [*Constr. Approx.* 3 (1987), No.3, 265-280] and Lou Yuanren [*Approx. Theory Appl.* 6 (1990), No.1, 46-64.]

In Chapter three some exact results are given by considering least square approximating polynomials to generalise the Walsh's result. Let $P_{n-1,r}(z;f)$ be the polynomial which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{q_n-1} |Q^{\nu}_{n-1,r}(\omega^k) - f^{\nu}(\omega^k)|^2, \qquad \omega^{q_n} = 1$$

over polynomials Q_{n-1} of degree $\leq n-1$, where $q_n = mn + c$ with $m \geq 1, 0 \leq c < m$ and $r \geq 1$, integers. We compare this polynomial with the polynomials obtained from power series expansion of $f \in A_\rho$ and obtain some exact results. The behaviour of the sequence of differences is also studied outside its region of convergence. We have succeeded in obtaining the results by considering n^{th} roots of α^n, $|\alpha| < \rho$, which generalise the results for n^{th} roots of unity given in the same chapter. Let $|\alpha| < \rho$ and $|\beta| < \rho$ be two arbitrary points, and let $f \in A_\rho$. Further, we assume $q_n \geq n, s_n \geq q_n$ and $t_n \geq s_n$, sequences of positive integers, satisfying some conditions. Let $P_{n-1,r}(z,\alpha,f)$ is the polynomial which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{q_n-1} |Q^{(\nu)}_{n-1}(\omega^k) - f^{(\nu)}(\omega^k)|^2, \qquad \omega^{q_n} = \alpha^{q_n}$$

over all polynomials $Q_{n-1} \in \Pi_{n-1}$. Similarly let $P_{s_n-1,r}(z,\beta,f)$ is the polynomial which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{t_n-1} |Q^{(\nu)}_{s_n-1}(\omega^k) - f^{(\nu)}(\omega^k)|^2, \qquad \omega^{t_n} = \beta^{t_n}$$

over all polynomials $Q_{s_n-1} \in \Pi_{s_n-1}$.

Here we study the sequence $\{P_{n-1,r}(z,\alpha;f) - P_{n-1,r}(z,\alpha;P_{s_n-1,1}(z,\beta;f))\}$ and obtain an exact result. Results of this chapter extend a result of A.S.Cavaretta, H.P.Dixit and A.Sharma [*Resultate Math* 7 (1984), No.2, 154-163] and generalise a result of M.P.Stojanova [*Math. Balkanica (N.S.)* 3 (1989), No.2, 149-171] and as a particular case they give results of Totik [ibid] and Ivanov & Sharma [ibid].

In Chapter four we are able to extend a few results of Chapter two by considering the average of the least square approximating polynomials. For positive integers m and n set

$\omega_{s,k} = exp[\frac{2\pi\imath}{mn}(km+s)]$, for $k = 0, \ldots, n-1$ and $s = 0, \ldots, m-1$. Here we study the polynomial

$$G_{n-1,r}(z;f) = \frac{1}{m}\sum_{s=0}^{m-1} G^s_{n-1,r}(z;f),$$

where for each $s = 0, \ldots, m-1$, $G^s_{n-1,r}(z;f)$ is the polynomial of degree $n-1$ which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{n-1} |Q^\nu_{n-1}(\omega_{s,k}) - f^\nu(\omega_{s,k})|^2$$

over polynomials Q_{n-1} of degree $\leq n-1$, where r is a fixed positive integer. The results of this chapter for n^{th} roots of unity are generalised for n^{th} roots of α^n. Let $|\alpha| < \rho$ and $|\gamma| < \rho$ be two arbitrary points, and let $f \in A_\rho$. Let $G_{n-1,r}(z,\alpha;f)$ is the polynomial

$$G_{n-1,r}(z,\alpha;f) = \frac{1}{m}\sum_{s=0}^{m-1} G^s_{n-1,r}(z,\alpha;f)$$

where $G^s_{n-1,r}(z,\alpha;f)$ is polynomial which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{n-1} |Q^{(\nu)}_{n-1}(\alpha\omega_{s,k}) - f^{(\nu)}(\alpha\omega_{s,k})|^2,$$

over all polynomials $Q_{n-1} \in \Pi_{n-1}$. Further, we assume $d_n \geq n$, a sequence of positive integers, satisfying some condition. For b a fixed positive integer let $\eta_{q,k} = exp[\frac{2\pi\imath}{db}(bk + q),]$ $q = 0, \ldots, b-1,\ k = 0, \ldots, d-1$. Let $G_{d_n-1,r}(z,\gamma;f)$ is the polynomial

$$G_{d_n-1,r}(z,\alpha;f) = \frac{1}{b}\sum_{q=0}^{b-1} G^q_{d_n-1,r}(z,\gamma;f)$$

where $G^q_{d_n-1,r}(z,\gamma;f)$ is polynomial which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{d_n-1} |Q^{(\nu)}_{d_n-1}(\gamma\eta_{q,k}) - f^{(\nu)}(\gamma\eta_{q,k})|^2,$$

over all polynomials $Q_{s-1} \in \Pi_{s-1}$.

We study the sequence $\{G_{n-1,r}(z,\alpha;f) - G_{n-1,r}(z,\alpha;G_{d-1,1}(z,\gamma;f))\}$ and obtain some exact results. In a special case the last result gives a result of M.P.Stojanova [ibid].

Chapter five incorporates the results for Hermite interpolating polynomials. Here quantitative estimates are obtained for the growth of the derivatives of the sequence of differences of two polynomials. Results obtained here extend the results of Lou Yuanren (obtained for Lagrange interpolation) [ibid] to Hermite interpolation and, in a particular case, they generalise the results of Ivanov and Sharma [*J. Approx. Theory (Beijing)* 2 (1986), No.1, 47-64] given for Hermite interpolation.

Chapter six is devoted to the study of polynomial interpolants in z, z^{-1}. We obtain two sequences of differences of polynomials, one in z and the other in z^{-1}. For each sequence of ordered pairs $\{(m_\iota, n_\iota)\}$ of non-negative integers, satisfying some condition let, $q_\iota \geq (m_\iota + n_\iota)$. For any f in A_ρ, let $L_{q_\iota - 1}(z; f)$ be the Lagrange interpolant of $z^{n_\iota} f(z)$ in $\Pi_{q_\iota - 1}$ at the $q_\iota{}^{th}$ roots of unity, then $A_{m_\iota + n_\iota - 1}(z; f) = S_{m_\iota + n_\iota - 1}(z; L_{q_\iota - 1}(z; f))$ where $S_{n-1}(z; g)$ denotes the $(n-1)^{th}$ partial sum of the power series of $g(z)$. Thus $z^{-n_\iota} A_{(m_\iota + n_\iota - 1)}(z; f)$ can be uniquely expressed as the sum of a polynomial in $\Pi_{m_\iota - 1}$ in the variable z and a polynomial in Π_{n_ι} in the variable z^{-1}. We compare the two sequences of polynomials by appropiate polynomials obtained from the power series expansion of f. Results here extend a result of Cavaretta et al [*Resultate Math.* 3 (1980), No.2,621-628.] Further, as a special case, the results for polynomials in z give results of V.Totik [ibid] and K.G.Ivanov & A.Sharma [ibid].

Finally, in Chapter seven, we consider functions analytic in an ellipse and hence represented by the Chebyshev sereis. We obtain quantitative estimates for the sequence of differences of two polynomials which are averages of polynomials associated with interpolating polynomials to a function at the zeros and extremas of the Chebyshev polynomials. Particular cases of these results give seperate results for the polynomials associated zeros and extremas of Chebyshev polynomials. Results of this chapter extend a result of T.J.Rivlin [*J. Approx. Theory* 36 (1982), N0.4, 334-345.]

Chapter 1

INTRODUCTION

1.1 We know that, at every point outside the circle of convergence of a power series, the series is divergent. But if, instead of considering the whole sequence of partial sums of the series, we a consider particular sequence of these sums, it is sometimes possible to obtain a convergent sequence. A power series which has a sequence of partial sums convergent outside the circle of convergence of the series is said to be 'overconvergent '

Let

$$g(z) = \sum_{n=1}^{\infty} \frac{\{z(1-z)\}^{4^n}}{s_n},$$

where s_n is the maximum coefficient in the polynomial $\{z(1-z)\}^{4^n}$. Then in each of the polynomials $\frac{\{z(1-z)\}^{4^n}}{s_n}$ the moduli of the coefficients do not exceed 1, and one of them is actually equal to 1. Also the highest term in this polynomial is of degree 2.4^n, whereas the lowest term in the next polynomial is of degree 4^{n+1}. Hence if we expand $g(z)$ in powers of z, $g(z) = \sum_{n=0}^{\infty} a_n z^n$, each term is a single term of one of the above polynomials. The radius of convergence of this series is 1, since $|a_n| \leq 1$ for all n, while $a_n = 1$ for an infinity of values of n.

In particular, the above series of polynomials is convergent for $|z| < 1$. But, since it is formally unchanged by the substitution $z = 1 - w$, it is also convergent for $|w| < 1$, i e , for $|1 - z| < 1$. The special sequence of partial sums obtained by taking each polynomial as a whole is therefore convergent in a region which lies partly outside the unit cirlcle.

Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be analytic in $|z| < \rho$ but not so in $|z| \leq \rho$. We shall denote the class of such functions by $A_\rho(\rho > 1)$. Let $L_{n-1}(z; f)$ denote the Lagrange interpolant to $f \in A_\rho$, of degree $(n-1)$ on the n^{th} roots of unity ; that is

$$L_{n-1}(\omega^k; f) = f(\omega^k), \qquad \omega^n = 1,\ k = 0, \ldots, n-1,$$

and let

$$P_{n-1}(z; f) = \sum_{k=0}^{n-1} a_k z^k$$

be the n^{th} partial sum of the power series expansion of $f(z)$ in $|z| < \rho$. It is known that both $L_{n-1}(z; f)$ [47] and $P_{n-1}(z; f)$ converge to $f(z)$ in $|z| < \rho$. Around 1931, Walsh [58] observed that the difference $L_{n-1}(z; f) - P_{n-1}(z; f)$ converges to zero in $|z| < \rho^2$, and that the convergence is uniform and geometric on any compact subset of $|z| < \rho^2$. This theorem on overconvergence is striking for its simplicity, directness and beauty and it is surprising that it remained unnoticed for almost half a century.

An example of D.J.Newman [12] shows that if the Lagrange interpolating polynomial is replaced by the best uniform approximating polynomial $L^1_{n-1}(z; f)$ of degree $n-1$ to $f \in A_\rho$ in $|z| \leq 1$, then we can find an $f_0 \in A_\rho$ such that $L^1_{n-1}(z; f_0) - P_{n-1}(z; f_0)$ converges to zero only for $|z| < \rho$. That is to say, for better approximating polynomials the domain of overconvergence shrinks.

In 1980 a straightforward extension of Walsh's result was obtained by Cavaretta et al [12] by comparing the interpolating polynomial $L_{n-1}(z; f)$ to polynomials determined from the power series expansion of f :

Theorem 1.1.1 *If for any integer $l \geq 1$, we set*

$$\Delta_{n-1,l}(z; f) = L_{n-1}(z; f) - Q_{n-1,l}(z; f),$$

where

$$Q_{n-1,l}(z; f) = \sum_{j=0}^{l-1} P_{n-1,j}(z; f) := \sum_{j=0}^{l-1} \sum_{k=0}^{n-1} a_{k+nj} z^k, \tag{1.1.1}$$

then

$$\lim_{n \to \infty} \Delta_{n-1,l}(z; f) = 0, \qquad \forall\ |z| < \rho^{1+l}. \tag{1.1.2}$$

The convergence in (1.1.2) is uniform and geometric on all compact subsets of the region $|z| < \rho^{l+1}$. Moreover, the result is the best possible in the sense that for any point z_0 with $|z_0| = \rho^{l+1}$, there is a function $f \in A_\rho$ for which (1.1.2) does not hold when $z = z_0$.

This theorem gave the idea to researchers of extending the Walsh theorem in various other directions. A brief skech of which we present now.

Throughout this chapter, unless otherwise stated, $\rho > 1$, and $f \in A_\rho$ always. Also l is a positive integer everywhere and Π_n denotes the collection of polynomials of degree at most n whereas C denotes the whole complex plane.

1.2 It may be noted that , *no* sharpness assertions are made in Walsh's theorem for *arbitrary* functions $f \in A_\rho$; in particular, no statement is made on the behaviour of the sequence $\{L_{n-1}(z; f) - P_{n-1}(z; f)\}_{n=1}^{\infty}$ in $|z| > \rho^2$. Saff and Varga [42] were the first to raise the question whether the difference $\Delta_{n-1,l}(z; f)$ can be bounded at some points outside the circle $|z| = \rho^{1+l}$. They proved [42] in 1983 that the sequence $\Delta_{n-1,l}(z; f)$ can be bounded in at most l distinct points in $|z| > \rho^{1+l}$, and moreover , given any l distinct points $\{z_j\}_{j=1}^{l}$ with $\rho^{1+l} < |z_j| < \rho^{2+l}$, $(j = 1, \ldots, l)$ there exists a function $f \in A_\rho$ such that

$$\Delta_{n-1,l}(z_j; f) \to 0 \quad as \ n \to \infty \quad (j = 1, \ldots, l).$$

The restriction on $|z_j|$ imposed above was later removed by Hermann [17]. He considered s and L positive integers with $s \leq l < L$ and $\{\eta_k\}_{k=1}^{s}$ distinct points with $\rho^{l+1} < |\eta_k| < \rho^{L+1}$, $(k = 1, 2, \ldots, s)$. Further, let $\phi \in A_{\rho^r}$ and $\psi(z)$ be analytic in $|z| \leq \alpha_s^{1/(l+1)}$, where r is least common multiple of $\{l+1, l+2, \ldots, L\}$ and $\alpha_s = \max_{1 \leq k \leq s} |\eta_k|$. With these notations Hermann [17] proved that if $\omega_s(z) = \Pi_{k=1}^{s}(z - \eta_k)$ and $f(z) = \omega_s(z)\phi(z^r) + \psi(z)$, then $f \in A_\rho$ and

$$\lim_{n\to\infty} \Delta_{n-1,l}(\eta_k; f) = 0 \quad (k = 1, \ldots, s).$$

In the direction of Saff and Varga's above result, further work was done by Lou Yuanren [26,27].

In 1986 , V.Totik [56] gave quantitative estimates for $\Delta_{n-1,l}(z; f)$ which supplement

and make more precise the sharpness result of Saff and Varga [42] above. Set

$$f_l(R) = \overline{\lim_{n\to\infty}}\{\max_{|z|=R} \Delta_{n-1,l}(z;f)|\}^{1/n}.$$

Using the identity

$$\Delta_{n-1,l}(z;f) = \sum_{k=0}^{n-1}\sum_{j=l}^{\infty} a_{k+nj}z^k = L_{n-1}(z;g_l)$$

where $g_l(z) = \sum_{k=ln}^{\infty} a_k z^k$, V.Totik proved that

$$f_l(R) = K_l(R,\rho), \qquad R > 0 \tag{1.2.1}$$

where

$$K_l(|z|,\rho) = \begin{cases} |z|/\rho^{l+1} & \text{if } |z| \geq \rho, \\ 1/\rho^l & \text{if } 0 \leq |z| \leq \rho, \end{cases}$$

which improves the above result of Cavaretta, Sharma and Varga [12] and as a corollary, he showed that if f is analytic in $|z| \leq 1$ and $f_l(R) = K_l(R,\rho)$ for some $R > 0, \rho > 1$, then f is analytic in $|z| < \rho$.

Next, considering the pointwise behaviour of $\Delta_{n-1,l}(z;f)$ Totik gave the exact form of Saff and Varga result [42]. If we set

$$B_l(z;f) = \overline{\lim_{n\to\infty}}|\Delta_{n-1,l}(z;f)|^{1/n}$$

then from (1.2.1) we have $B_l(z;f) \leq K_l(|z|,\rho)$. However it is not clear whether there are points for which $B_l(z;f) = K_l(|z|,\rho)$. In order to examine this situation, we put

$$\delta_{l,\rho}(f) := \{z|B_l(z;f) < K_l(|z|,\rho)\}, \qquad f \in A_\rho,\ \rho > 1.$$

Then Theorem 3 of Totik [56] can be stated as follows : If $f \in A_\rho, \rho > 1$ and l is any fixed positive integer, then

$$|\delta_{l,\rho}(f) \cap \{z||z| > \rho\}| \leq l$$

and

$$|\delta_{l,\rho}(f) \cap \{z|0 < |z| < \rho\}| \leq l-1$$

where $|S|$ denotes the cardinality of the set S.

Totik also showed (Theorem 4, [56]) that for any l points $\{z_j\}_{j=1}^{l}$ with moduli $> \rho$, there exists an $f \in A_\rho$ such that $z_j \in \delta_{l,\rho}(f)$, $j = 1,\ldots,l$. On the other hand, for

any $l-1$ points $\{z_j\}_{j=1}^{l-1}$ with moduli $< \rho$ and > 0, there exists an $f \in A_\rho$ such that $z_j \in \delta_{l,\rho}(f)$, $j = 1, \ldots, l-1$.

These results show that equality of $B_l(z; f)$ and $K_l(|z|, \rho)$ holds for all but a few exceptional points. But it is not clear whether the l points in $\delta_{l,\rho}(f) \cap \{z||z| > \rho\}$ and the $l-1$ points in $\delta_{l,\rho}(f) \cap \{z|0 < |z| < \rho\}$ can exist at the same time. In order to settle this problem K.G.Ivanov and Sharma [19] introduced the notion of an (l, ρ)- distinguished set.

A set Z is an (l, ρ)- distinguished set if for some $f \in A_\rho, B_l(z; f) < K_l(|z|, \rho)$ for $z \in Z$.

The order of convergence (or divergence) at a point of an (l, ρ)- distinguished set is better than at other points in its neighbourhood for some $f \in A_\rho$. Their Theorem 1 gives a criterion to determine whether Z is an (l, ρ)- distinguished set or not, when $Z = \{z_j\}_{j=1}^{s}$ is such that $|z_j| < \rho, j = 1, 2, \ldots, \mu$ and $|z_j| > \rho, j = \mu+1, \ldots, s$. In order to state the result, they defined the matrices X, Y and $M = M(X, Y)$ as follows

$$X := \begin{pmatrix} 1 & z_1 & \cdots & z_1^{l-1} \\ 1 & z_2 & \cdots & z_2^{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_\mu & \cdots & z_\mu^{l-1} \end{pmatrix}, Y := \begin{pmatrix} 1 & z_{\mu+1} & \cdots & z_{\mu+1}^{l} \\ 1 & z_{\mu+2} & \cdots & z_{\mu+2}^{l} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_s & \cdots & z_s^{l} \end{pmatrix}$$

The matrices X and Y are of order $\mu \times l$ and $(s-\mu) \times (l+1)$ respectively. Let

$$M := M(X, Y) := \begin{pmatrix} X & & & \\ & X & & \\ & & \ddots & \\ & & & X \\ Y & & & \\ & Y & & \\ & & \ddots & \\ & & & Y \end{pmatrix} \tag{1.2.2}$$

where X is repeated diagonally $l+1$ times and Y is repeated l times so that M has $sl + \mu$ rows and $l(l+1)$ columns. With the above notations Ivanov and Sharma [19] showed that the set Z is (l, ρ) distinguished iff $rank\ M < l\,(l+1)$. As a corollary of this result it follows that if either $\mu \geq l$ or $s - \mu \geq l+1$, then Z is not an (l, ρ)- distinguished set and if $\mu < s \leq l$ or $\mu = s < l$, then Z is an (l, ρ)- distinguished set, which are Totik's results

[56] given earlier. Totik [56] does not say anything about points on $|z| = \rho$. Ivanov and Sharma [19] considered the behaviour of $B_l(z; f)$ when $z \in \Gamma_\rho := \{z||z| = \rho\}$. From (1.2.1) it follows that $B_l(z; f) \leq \rho^{-l}$, when $z \in \Gamma_\rho$. Let $\gamma_l(f) := \delta_{l,\rho}(f) \cap \Gamma_\rho$, that is the set of points on Γ_ρ for which the strict inequality $B_l(z; f) < \rho^{-l}$ holds. Then, $\gamma_l^c(f) := \Gamma_\rho \backslash \gamma_l(f)$ will denote the complement of $\gamma_l(f)$ on Γ_ρ.

If $f_0(z) = (1 - z\rho^{-1})^{-1}$, then $\gamma_l(f_0) = \phi$, but it may happen that $\gamma_l(f)$ is even dense in Γ_ρ for some f. In fact, as shown by Ivanov and Sharma [19] $\gamma_1^c(f)$ is always dense in Γ_ρ The structure of $\gamma_l(f)$ is related to the dependence of $B_l(z; f)$ on z. To this effect Ivanov and Sharma [19] proved that any set of $l + 1$ points on Γ_ρ is an (l, ρ)- distinguished set.

It is natural to ask if we can add another point z_0 from Γ_ρ to Z so that the augmented set, $\{z_0\} \cup Z$, still remains an (l, ρ)- set. Ivanov and Sharma [19] gave answer to this question only for $l = 1$. In order to do so, set $U(Z)$ and $U^\star(Z)$ for a set Z of $l + 1$ distinct points $\{z_j\}_{j=1}^{l+1}$ on Γ_ρ as

$$U(z) = \{z_0 | z_0 \in \Gamma_\rho \backslash Z \text{ and } B_l(z_j; f) < \rho^{-l}, j = 0, 1, \ldots, l + 1, \text{ for some } f \in A_\rho\}.$$

$U^\star(Z) = \Gamma_\rho \backslash \{U(Z) \cup Z\}$. Then we have for $l = 1$, the result that if $z_j = \rho e^{(2\pi i \alpha_j)}, j = 1, 2$ and if $\alpha_2 - \alpha_1 = \alpha$ is an algebraic number of degree $\nu \geq 2$, then $U(z_1, z_2) = \phi$. As a consequence of which we have for any two distinct points $z_1, z_2 \in \Gamma_\rho$, the set $U^\star(z_1, z_2)$ is dense in Γ_ρ and, for each $f \in A_\rho$, the set $\gamma_1^c(f)$ is dense in Γ_ρ.

Ivanov and Sharma [19] also gave a sufficient condition for a finite set Z to be (l, ρ) distinguished proving that for any finite $Z = \{z_j\}_{j=1}^s$ on Γ_ρ, where $z_j = \rho e^{(2\pi i \alpha_j)}, \alpha_j$ a rational, there exists an $f \in A_\rho$ such that $Z \subset \gamma_1(f)$.

The (l, ρ) distinguished sets were further studied in [22].

What is so peculiar about the roots of unity ? This question has arisen in the minds of many. The first one to raise this question was Baishanski [2]. Later, Szabados and Varga [53] considered a triangular matrix Z whose n^{th} row contains the entries $\{z_{k,n}\}_{k=1}^n$ with $0 \leq |z_{k,n}| < \rho$. Associated with the nth row of Z is the monic polynomial of degree n:

$$w_n(u) = w_n(u, Z) := \prod_{k=1}^{n} (u - z_{k,n}), \qquad n \geq 1.$$

Let

$$\gamma_n(\rho, Z) := \text{modulus of the first non-zero term of } \begin{cases} w_n(\rho, Z), & \text{if } l > 1, \\ w_n(\rho, Z) - \rho^n, & \text{if } l = 1. \end{cases}$$

Note that since $w_n(u, Z)$ is monic, then $\gamma_n(\rho, Z)$ is well defined for all $l > 1$ and $\gamma_n(\rho, Z) > 0$ for all $n \geq 1$. However, if $w_n(u, Z) = u^n$ and if $l = 1$, then all the terms of $w_n(\rho, Z) - \rho^n$ are zero, and $\gamma_n(\rho, Z)$ is defined to be zero in this case. Thus Szabados and Varga [53] made the assumption about the matrix Z that

$$\mu = \mu(\rho, Z) := \overline{\lim_{n\to\infty}} \gamma_n^{1/n}(\rho, Z) \geq 1.$$

Let $L_{n-1}(z; Z; f)$ be the Lagrange interpolating polynomial to f of degree $\leq n-1$ based on the nodes determined by the n^{th} row of the infinite triangular matrix $\{z_{k,n}\}, k = 1, \ldots, n,\ \ n \geq 1$. With the above definitions and assumptions they proved that for each complex number $\hat{z}$ with $|\hat{z}| > \rho^{l+1}/\mu$, there is an $\hat{f}$ in A_ρ for which the sequence $\{L_{n-1}(\hat{z}; Z; \hat{f}) - Q_{n-1,l}(\hat{z}; \hat{f})\}_{n=1}^{\infty}$ is unbounded, where $Q_{n-1,l}(z; f)$ is given by (1.1.1). In addition, there holds

$$\Delta_l(R, \rho, Z) \geq \Delta_l(R, \rho, E), \qquad \forall\ R > \rho, \tag{1.2.3}$$

where

$$\Delta_l(R, \rho, Z) = \sup_{f\in A_\rho} \overline{\lim_{n\to\infty}} \{\max_{|z|=R} |L_{n-1}(z; Z; f) - Q_{n-1,l}(z; f)|\}^{\frac{1}{n}}, \quad R > \rho$$

and E is the matrix Z with $z_{k,n} = exp(2k\pi i/n)(k = 1, 2, \ldots, n)$. Equality holds in (1.2.3) for all $R > \rho$ if for some positive integer l the matrix Z satisfies

$$|z_{k,n} - exp(2k\pi i/n)| \leq 1/\rho^{ln} \qquad (k = 1, \ldots, n; n \geq 1).$$

In [54] , they made the stronger hypothesis for Z that there exists a real number $\hat{\rho}$ with $1 \leq \hat{\rho} < \rho$ for which

$$1 \leq |z_{k,n}| \leq \hat{\rho} < \rho\ \ (k = 1, \ldots, n;\ \ n = 1, 2, \ldots),$$

and they wanted to determine the domain in the complex plane for which the sequence $\{L_{n-1}(z; Z; f) - Q_{n-1,l}(z; f)\}$ converges geometrically to zero for all $f \in A_\rho$. The answer to this question led them to study the functions $G_l(z, R)$ and $\hat{G}_l(z, \rho)$ where

$$\begin{aligned} G_l(z, R) &= G_l(z, Z, R) \\ &= \overline{\lim_{n\to\infty}} \max_{|t|=R} |(1 - t^{-ln})(\frac{z^n - 1}{t^n - 1}) - \frac{\omega_n(z, Z)}{\omega_n(t; Z)}|^{1/n} \end{aligned}$$

and

$$\hat{G}_l(z,\rho) = \inf_{\hat{\rho}<R<\rho} G_l(z,R),$$

where $\omega_n(z,Z) = \prod_{k=1}^{n}(z - z_{k,n})$. They established that [54] for any complex number $z \neq 1$ and any positive integer l, there holds

$$G_l(z,R) \geq \frac{max\{|z|,1\}}{R^{l+1}}, \qquad R > \hat{\rho}$$

and

$$\hat{G}_l(z,\rho) \geq \sup_{f\in A_\rho} \overline{\lim_{n\to\infty}} |L_{n-1}(z;Z;f) - Q_{n-1,l}(z;f)|^{1/n} \geq G_l(z,\rho).$$

In the end Szabados and Varga raisd three open questions :

1. Is $G_l(z,\rho) = \hat{G}_l(z,\rho)$?

2. If yes, then @ $:= \{z : G_l(z,\rho) = 1\}$ divides the complex plane into sets where either one has geometric convergence to zero for all f in A_ρ or unboundedness of the sequence $\{L_{n-1}(z;Z;f) - Q_{n-1,l}(z;f)\}$ for some $f \in A_\rho$. What does @ $:= \{z : G_l(z,\rho) = 1\}$ look like ?

3. In general, one would not suspect that @ is a circle, even though this is the case for all examples treated in the literature. Can one construct cases (i.e. matrices Z) where @ is not a circle ?

In [55] Totik gave affirmative answer to all the three questions by giving surprising results.

Recently J.Szabados [52] proved a converse result of Theorem 1.1.1 showing that if $f(z)$ is analytic in $|z| < 1$ and continuos in $|z| \leq 1$, and if $\{\Delta_{n-1,l}(z;f)\}_{n=1}^{\infty}$ is uniformly bounded on every closed subset of $|z| < \rho^{1+l}$, then $f(z)$ is analytic in $|z| < \rho$.

The use of Saff and Varga result [42] is basic in the proof this result. Szabados asks whether a proof of the converse result can be given without using the sharpness result of Saff-Varga. This question was settled by Ivanov and Sharma [21]. They considered two entities - a null sequence $\{a_k\}_{k=0}^{\infty}$ of real or complex numbers and a function f defined on all the roots of unity, that is, on $U = \cup_{n=1}^{\infty} U_n$, where $U_n = \{exp(2k\pi i/n),\ k = 0,1,\ldots,n-1\}$.

For any fixed integer $l \geq 1$ define

$$\Delta_{n-1,l}(z;\{a_k\}_0^\infty;f) := L_{n-1}(z;f) - \sum_{k=0}^{n-1}\sum_{j=0}^{l-1} a_{k+nj}z^k.$$

If $A(\rho)$ denotes the class of functions analytic in $|z| < \rho$, then they proved that the sequence $\{\Delta_{n-1,l}(z;\{a_k\}_0^\infty;f)\}_{n=1}^\infty$ is uniformly bounded on compact subsets of $|z| < \rho^{l+1}$ iff

(a) the function $g(z) := \sum_{k=0}^\infty a_k z^k \in A(\rho)$ and

(b) there exists a function $h(z) \in A(\rho^{l+1})$ such that

$$(g+h)(z) = f(z), \qquad z \in U.$$

1.3 In 1982 T.J.Rivlin [39] extended the result of Walsh in another direction by considering least square apprroximation to functions by polynomials of degree n on the q^{th} roots of unity. He considered $q = mn + c$, where m, c are positive integers, and determined the polynomial $P_{n-1,q}(z;f) \in \Pi_{n-1}$ which minimizes

$$\sum_{k=0}^{q-1} |f(\omega^k) - p(\omega^k)|^2, \qquad \omega^q = 1 \tag{1.3.1}$$

over all polynomials $p \in \Pi_{n-1}$. He proved [39] that if we set

$$\Delta_{n-1,q,l}(z;f) = P_{n-1,q}(z;f) - \sum_{j=0}^{l-1}\sum_{k=0}^{n-1} a_{k+qj}z^k,$$

then

$$\lim_{n\to\infty} \Delta_{n-1,q,l}(z;f) = 0, \qquad \forall \ |z| < \rho^{1+lm}.$$

Rivlin proved this for $l = 1$, but the general case for $l > 1$ follows easily. Following the lines of Totik [56], quantitative estimates have been found for $\Delta_{n-1,q,l}(z;f)$ also by Ivanov and Sharma [20]. If we set

$$\overline{B}_{l,m}(R;f) = \overline{\lim_{n\to\infty}} \sup_{|z|=R} |\Delta_{n-1,q,l}(z;f)|^{1/n}$$

and

$$B_{l,m}(z;f) = \overline{\lim_{n\to\infty}} |\Delta_{n-1,q,l}(z;f)|^{1/n}$$

then, Ivanov and Sharma [20] proved that

$$\overline{B}_{l,m}(R;f) = K_{l,m}(R,\rho), \qquad R > 0, \tag{1.3.2}$$

where

$$K_{l,m}(|z|,\rho) := \begin{cases} \rho^{-lm}, & 0 \le |z| \le \rho \\ |z|\rho^{-1-lm}, & \rho \le |z|. \end{cases} \tag{1.3.3}$$

From the definitions and (1.3.2) it is clear that $B_{l,m}(z;f) \le K_{l,m}(|z|,\rho)$. In this case we say that a set $Z \subset C$ is (l,m,ρ) - distinguished if there exists an $f \in A_\rho$ such that

$$B_{l,m}(z;f) < K_{l,m}(|z|,\rho) \qquad \text{for every } \ z \in Z.$$

If $Z = \{z_j\}_{j=1}^{s}$ with $|z_j| < \rho, j = 1,\ldots,\mu$ and $|z_j| > \rho, j = \mu+1,\ldots,s$, we set

$$X = (z_i^j)_{1\le i\le \mu,\ 0\le j\le lm-1},\ \ Y = (z_i^j)_{\mu+1\le i\le s,\ 0\le j\le lm}.$$

Let $M_1 = M(X,Y)$ denote a matrix analogous to (1.2.2), with X repeated $lm+1$ times and Y repeated lm times. Then Ivanov and Sharma [20] established that Z is (l,m,ρ)-distinguished iff

$$rank\ M_1 < lm(lm+1),$$

from which we immediatly derive that if either $\mu \ge lm$ or $s-\mu \ge lm+1$, then Z is not an (l,m,ρ)- distinguished set and if $\mu < s \le lm$ or $\mu = s < lm$, then Z is an (l,m,ρ)-distinguished set.

Next, considering the case when all points of Z lie on $|z| = \rho$, it is proved [20] that any $lm+1$ points on $|z| = \rho$ form a an (l,m,ρ)- distinguished set.

1.4 In 1980 Cavaretta et al [12] extended Walsh's result to Hermite interpolation as well. For any positive integer r, let $h_{rn-1}(z;f)$ be the Hermite interpolant to $f(z)$ on the zeros of $(z^n-1)^r$.That is,

$$h_{rn-1}^{\nu}(\omega^k) = f^{\nu}(\omega^k),$$

where $\omega^n = 1, \nu = 0,1,\ldots,r-1$ and $k = 0,\ldots,n-1$. Let

$$\beta_{j,r}(z) = \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z-1)^k, \qquad j \in N.$$

Set

$$H_{rn-1,0}(z;f) = \sum_{k=0}^{rn-1} a_k z^k,$$

$$H_{rn-1,j}(z;f) = \beta_{j,r}(z^n) \sum_{k=0}^{n-1} a_{k+n(r+j-1)} z^k, \ \ j = 1,2,\ldots$$

and

$$\Delta_{rn-1,l}(z;f) := h_{rn-1}(z;f) - \sum_{j=0}^{l-1} H_{rn-1,j}(z;f).$$

Then according to their direct theorem in the Hermite case [12],

$$\lim_{n\to\infty} \Delta_{rn-1,l}(z;f) = 0, \qquad \forall \ |z| < \rho^{1+\frac{l}{r}}.$$

Giving a sharpness result for Hermite interpolation Saff and Varga [42] proved that the sequence $\Delta_{rn-1,l}(z;f)$ can be bounded in at most $r+l-1$ distinct points in $|z| > \rho^{1+(l/r)}$, furthermore , given any $r+l-1$ distinct points $\{z_j\}_{j=1}^{r+l-1}$ in the annulus $\rho^{1+(l/r)} < |z| < min\{\rho^{l+2}, \rho^{1+l/(r-1)}\}$, there exists a function $f \in A_\rho$ such that $\Delta_{rn-1,l}(z_j;f) \to 0 \ \ as \ n \to \infty \ \ (j = 1, \ldots, r+l-1)$.

Cavaretta et al [14] also considered mixed Hermite interpolation and least square approximation to extend Walsh's result.

For the quantitative estimates in case of Hermite interpolation , we set

$$\overline{D}_{l,r}(R;f) = \overline{\lim_{n\to\infty}} \sup_{|z|=R} |\Delta_{rn-1,l}(z;f)|^{1/rn}$$

and

$$D_{l,r}(z;f) = \overline{\lim_{n\to\infty}} |\Delta_{rn-1,l}(z;f)|^{1/rn}.$$

As an analogue of Theorem 1 of Totik [56], Ivanov and Sharma [20] proved that

$$\overline{D}_{l,r}(R;f) = K^1_{l,r}(R,\rho), \qquad R > 0, \tag{1.4.1}$$

where

$$K^1_{l,r}(z,\rho) := \begin{cases} \rho^{-1-\frac{l-1}{r}}, & |z| \le 1 \\ |z|^{1-\frac{1}{r}}\rho^{-1-\frac{l-1}{r}}, & 1 \le |z| \le \rho \\ |z|\rho^{-1-\frac{l}{r}}, & \rho \le |z|. \end{cases}$$

From the definitions and (1.4.1) it is clear that $D_{l,r}(z;f) \le K^1_{l,r}(z,\rho)$. In this case we say that a set $Z \subset C$ is (l,r,ρ)- distinguished if there exists an $f \in A_\rho$ such that $D_{l,r}(z;f) < K^1_{l,r}(z,\rho)$ for every $z \in Z$. If $Z = \{z_j\}_{j=1}^{s}$ with $|z_j| < \rho, j = 1, \ldots, \mu$ and $|z_j| > \rho, j = \mu+1, \ldots, s$, we set

$$X = (z_i^j)_{1\le i\le\mu,\ 0\le j\le r+l-2}, \ \ Y = (z_i^j)_{\mu+1\le i\le s,\ 0\le j\le r+l-1}.$$

Let $M_2 = M(X,Y)$ denote a matrix analogous to (1.2.2), with X repeated $r+l-1$ times and Y repeated $r+l$ times. Then Ivanov and Sharma [20] established that Z is (l,r,ρ)-distinguished iff $rank\ M_1 < (r+l)(r+l-1)$, from which , we immediatly derive that if either $\mu \geq r+l-1$ or $s-\mu \geq r+l$, then Z is not an (l,r,ρ)- distinguished set and if $\mu < s \leq r+l-1$ or $\mu = s < r+l-1$, then Z is an (l,r,ρ)- distinguished set, which includes Theorem 2 in Saff and Varga [42].

Next, considering the case when all points of Z lie on $|z| = \rho$, it is proved that [20] any $l+r$ points on $|z| = \rho$ form an (l,r,ρ)- distinguished set.

In 1985 follwing the same idea as in Lagrange interpolation, Cavaretta et al [15] proved the converse result in case of Hermite interpolation. They showed that if f is analytic in $|z| < 1, f, f', \ldots, f^{r-1}$ be all continous on $|z| = 1$, and if $\{\Delta_{rn-1,l}(z;f)\}_{n=1}^{\infty}$ is uniformly bounded on every closed subset of $|z| < \rho^{1+\frac{l}{r}}$ then f is analytic in $|z| < \rho$.

1.5 The phenomenon of overconvergence in the result of Walsh was studied in a different manner by T.E.Price in 1985 who considered averages of interpolating polynomials. More specifically, let m and n be positive integers and let $\omega = e^{\frac{2\pi i}{mn}}$. Set $f_q(z) = f(z\omega^q)$, $q = 0, 1, \ldots, m-1$, and define the averages

$$A_{n-1,m}(z;f) = \frac{1}{m} \sum_{q=0}^{m-1} L_{n-1}(z\omega^{-q}; f_q)$$

and

$$A_{n-1,m,j}(z;f) = \frac{1}{m} \sum_{q=0}^{m-1} P_{n-1,j}(z\omega^{-q}; f_q) \qquad j = 0, 1, 2, \ldots,$$

where $P_{n-1,j}(z;f) = \sum_{k=0}^{n-1} a_{k+nj} z^k$. It is easy to see that

$$A_{n-1,j} = \begin{cases} P_{n-1,j} & \text{if } j = pm,\ p \geq 0, an integer \\ 0 & \text{otherwise} \end{cases}$$

Also for $0 \leq q \leq m-1$, $L_{n-1}(z\omega^{-q}; f_q)|_{z=\omega^{jm+q}} = f_q(\omega^{jm}) = f(\omega^{jm+q}), j = 0, 1, \ldots, n-1$, so that $L_{n-1}(z\omega^{-q}; f_q)$ may be considered as the Lagrange interpolant of f in the nodes $\{\omega^{jm+q}\}_{j=0}^{n-1}$. With these notations Price [33] showed that if β be the least positive integer such that $\beta m > l-1$, Then

$$\lim_{n\to\infty} \left(A_{n-1,m}(z;f) - \sum_{j=0}^{l-1} A_{n-1,m,j}(z;f) \right) = 0 \qquad \forall\ |z| < \rho_0 = \rho^{1+\beta m}$$

The convergence is uniform in each disc $|z| \leq Z < \rho_0$. Further the radius ρ_0 is the largest for which such a result holds. Note that the Walsh's theorem cited earlier is the case when $m = l = 1$. The proof of the theorem was obtained by the use of integral formulas for the various quantities involved.

In next theorem Price [33] stated that if f is also continous on the closed disc $|z| \leq \rho$, then the term in paranthesis in above result goes to zero for all z with $|z| \leq \rho_0$. He also extend his first result to the case of Hermite interpolation of order r (interpolation to f and its first $r-1$ derivatives at the roots of unity).

1.6 Another variation of the condition (1.3.1) was considered in [10] when it was replaced by

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{q-1} |f^{(\nu)}(\omega^k) - Q_n^{(\nu)}(\omega^k)|^2 \tag{1.6.1}$$

where $q = mn + c, \omega^q = 1$ and r is a fixed integer. The unique polynomial $P_{n,r}(z; f)$ which minimizes (1.6.1) over all polynomials $Q_n \in \Pi_{n-1}$ is given by

$$P_{n,r}(z; f) = \sum_{j=0}^{n-1} c_j z^j$$

where

$$c_j = \frac{1}{A_{0,j}(r)} \sum_{\lambda=0}^{\infty} A_{\lambda,j}(r) a_{j+\lambda q}, \qquad j = 0, 1, \ldots, n-1$$

and

$$A_{\lambda,j}(r) = \sum_{i=0}^{r-1} (j)_i (j+\lambda q)_i, \qquad (j)_i = j(j-1), \ldots, (j-i+1).$$

Set

$$S_{n,\lambda,r}(z; f) = \sum_{j=0}^{n-1} \frac{A_{\lambda,j}(r)}{A_{0,j}(r)} a_{j+\lambda q} z^j, \qquad \lambda = 0, 1, \ldots.$$

Then Cavaretta, Dikshit and Sharma [10] established that

$$\lim_{n\to\infty} \left\{ P_{n,r}(z; f) - \sum_{\lambda=0}^{l-1} S_{n,\lambda,r}(z; f) \right\} = 0 \qquad \forall \ |z| < \rho^{1+lm}.$$

Similar result is obtained for Hermite interpolation on replacing (1.6.1) by

$$\sum_{k=0}^{q-1}\sum_{\nu=r}^{r+1} |P_{rq+n}^{(\nu)}(\omega^k; f) - f^{(\nu)}(\omega^k)|^2. \tag{1.6.2}$$

Cavaretta, Dixit and Sharma studied a variation in Hermite and l_2 - approximation to determine $P_{rq+n}(z; f)$ of the form

$$P_{rq+n}(z; f) = h_{rq-1}(z; f) - (z^q - 1)^r Q_n(z), \qquad Q_n(z) \in \Pi_{n-1},$$

where $h_{rq-1}(z; f)$ denotes the Hermite interpolant to f and its first $r-1$ derivatives at the q^{th} roots of unity, by requiring that (1.6.2) is minimized. Following Saff and Varga [42] and Hermann [17] in the same paper Cavaretta, Dixit and Sharma have asked for the existence of function $f \in A_\rho$ for which the difference $\left\{P_{n,r}(z; f) - \sum_{\lambda=0}^{l-1} S_{n,\lambda,r}(z; f)\right\}$ tends to zero as $n \to \infty$ in at least l points lying outside the circle $|z| < \rho^{1+lm}$. For $r = 1$ they had given answer to this question in the affirmative but no comment is made for $r > 1$.

Recently, Juneja and Dua [24] obtained an exact form of Cavaretta, Dixit and Sharma's [10] result stated above for the Lagrange interpolation. They established that [24]

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |P_{n,r}(z; f) - \sum_{\lambda=0}^{l-1} S_{n,\lambda,r}(z; f)|^{1/n} = K_{l,m}(R, \rho), \qquad R > 0$$

where $K_{l,m}(|z|, \rho)$ is given by (1.3.3) .

1.7 In 1986 Lou Yuanren [25] gave a new direction in extensions of Walsh's theorem by considering interpolation in the roots of α^n, $|\alpha| < \rho$. Let $\alpha, \beta \in D_\rho$ where $D_\rho := \{z||z| < \rho\}$, and for any two positive integers m and $n (m > n)$, let $L_{n-1}(z, \alpha, f)$ and $L_{m-1}(z, \beta, f)$ denote the Lagrange interpolating polynomial to f on the zeros of $z^n - \alpha^n$ and $z^m - \beta^m$ respectively . Let $m = m_n = rn + q$, $s \le q/n < 1$ and $q/n = s + \mathcal{O}(\frac{1}{n})$ and set

$$\Delta_{n,m}^{\alpha,\beta}(z; f) := \{L_{n-1}(z, \alpha, f) - L_{n-1}(z, \alpha, L_{m-1}(z, \beta, f))\}.$$

Then Lou Yuanren [25] proved that for $\alpha \neq \beta$

$$\lim_{n\to\infty} \Delta_{n,m}^{\alpha,\beta}(z; f) = 0, \qquad \forall \ |z| < \sigma$$

where

$$\sigma := \rho / max\left\{\left(\frac{|\alpha|}{\rho}\right)^r, \left(\frac{|\beta|}{\rho}\right)^{r+s}\right\}.$$

When $\alpha = 1, \beta = 0$, and $m = rn, r = l$, the above result yeilds Theorem 1.1.1. Akhlagi. Jakimovski and Sharma [1] gave analogues of above result to mixed Lagrange interpolation and l_2 - approximation. They also established more precise result for the differences $\Delta_{n,m}^{\alpha,\beta}$ by showing that if $|\alpha/\rho|^r \neq |\beta/\rho|^{r+s}$ and for $s \neq 0$ if $|\alpha/\rho|^{r+1} \neq |\beta/\rho|^{r+s}$, then

$$\overline{\lim_{n\to\infty}} \{\max_{|z|=R} |\Delta_{n,m}^{\alpha,\beta}(z; f)|^{1/n}\} = K_\rho(R), \qquad R > 0, \tag{1.7.1}$$

where

$$K_\rho(|z|) = \begin{cases} (|z|/\rho) max\{|\alpha/\rho|^r, |\beta/\rho|^{r+s}\} & \text{for } \ |z| \ge \rho \\ max\{|\alpha/\rho|^{r+1}, |\alpha/\rho|^r (|z|/\rho)^s, |\beta/\rho|^{r+s}\}. & \text{for } \ 0 < |z| < \rho \end{cases}$$

In the special case $\alpha = 1, \beta = 0$ and $m = rn, r = l$, this reduces to Totik's theorem [56]. This result was further analysed by M.P.Stojanova [51].

From (1.7.1) we have the inequality

$$\overline{\lim_{n\to\infty}} |\Delta_{n,m}^{\alpha,\beta}(z; f)|^{1/n} \leq K_\rho(|z|).$$

If there is some function $f \in A_\rho$ such that

$$\overline{\lim_{n\to\infty}} |\Delta_{n,m}^{\alpha,\beta}(z; f)|^{1/n} < K_\rho(|z|)$$

is true for each $z \in Z$, we shall say that Z is $(\{\Delta_{n,m}^{\alpha,\beta}\}, \rho)$ - distinguished set.

It is clear that the number of points in some $(\{\Delta_{n,m}^{\alpha,\beta}\}, \rho)$ - distinguished set depends on the behaviour of the sequence $\{m_n\}_{n=0}^{\infty}$. Let us denote

$$\delta(\{m_n\}) = \overline{\lim_{n\to\infty}} (m_{n+1}^{*} - m_n^{*}),$$

where $\{m_n^*\}_{n=0}^{\infty}$ is the non-decreasing rearrangment of $\{m_n\}_{n=0}^{\infty}$.

Then M.P.Stojanova [50] proved

Theorem 1.7.1 [50] Let $m = m_n = rn + q, q = q_n = sn + \mathcal{O}(1), 0 \leq s < 1, q_n \geq 0$ and $\alpha, \beta \in D_\rho$, and let $\rho_1 = |\beta||\beta/\alpha|^{r/s}, \rho_2 = \rho|\alpha/\rho|^{1/s}$. Then the set $Z \subset \Omega$ is an $(\{\Delta_{n,m}^{\alpha,\beta}\}, \rho)$ - distinguished iff

$$(a) \qquad |Z| < \begin{cases} \delta(\{m_n\}) & \text{for } \Omega = D_\rho \\ \delta(\{m_n + n\}) & \text{for } \Omega = C \backslash D_\rho \end{cases}$$

in the case $|\frac{\beta}{\rho}|^{r+s} > |\frac{\alpha}{\rho}|^r$.

$$(b) \qquad |Z| < \begin{cases} \delta(\{m_n\}) \ \text{ for } \ \Omega = \begin{cases} D_\rho & \text{for } q_n = 0 \\ D_\rho \backslash \Gamma_{\rho_1} & \text{otherwise} \end{cases} \\ r+1 \qquad \text{for } \ \Omega = C \backslash \overline{D_\rho} \end{cases}$$

in the cases $|\frac{\alpha}{\rho}|^{r+1} < |\frac{\beta}{\rho}|^{r+s} < |\frac{\alpha}{\rho}|^r, s \neq 0$ and $|\frac{\beta}{\rho}|^r < |\frac{\alpha}{\rho}|^r, s = 0$.

$$(c) \qquad |Z| < \begin{cases} \delta(\{m_n\}) & \text{for } \Omega = D_\rho \backslash \overline{D_{\rho_2}} \\ r+1 & \text{for } \Omega = D_{\rho_2} \cup \{C \backslash \overline{D_\rho}\} \end{cases}$$

in the case $|\frac{\beta}{\rho}|^{r+s} < |\frac{\alpha}{\rho}|^{r+1}, s \neq 0$.

Since $\delta(\{rn\}) = r$ for $r > 0$, Theorem 1.7.1 gives corresponding result of [19]. Lou Yuanren [29] gave convergence results in this direction by considering Hermite interpolation.

1.8 In 1980 Cavaretta et al [12] gave some results for interpolation by considering polynomials in z and z^{-1} as well. For each ordered pair (m_i, n_i) of non-negative integers, and for any $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in A_ρ, let $q^{(m_i,n_i)}(z; f)$ be the Lagrange interpolant of $z^{n_i} f(z)$ in $\Pi_{m_i+n_i}$ at the $(m_i+n_i+1)^{th}$ roots of unity, then $z^{-n_i} q^{(m_i,n_i)}(z; f)$ can be uniquely expressed as the sum of a polynomial in Π_{m_i} in the variable z and a polynomial in Π_{n_i} in the variable z^{-1}, that is if $q^{(m_i,n_i)}(z; f) = \sum_{j=0}^{m_i+n_i} \alpha_j z^j$, then

$$z^{-n_i} q^{(m_i,n_i)}(z; f) = r_{m_i}^{(m_i,n_i)}(z; f) + s_{n_i}^{(m_i,n_i)}(z^{-1}; f),$$

where $r_{m_i}^{(m_i,n_i)}(z; f) = \sum_{j=0}^{m_i} \alpha_{j+n_i} z^j$ and $s_{n_i}^{(m_i,n_i)}(z^{-1}; f) = \sum_{j=0}^{n_i-1} \alpha_j z^{j-n_i}$. Now define

$$P_{m_i,n_i,j}(z; f) = \sum_{k=0}^{m_i} a_{j(m_i+n_i+1)+k} z^k, \qquad j \geq 0$$

and

$$Q_{n_i,m_i,j}(z^{-1}; f) = \sum_{k=0}^{n_i-1} a_{j(m_i+n_i+1)-n_i+k} z^{k-n_i}, \qquad j \geq 1$$

With the above notations generalizing a result of Walsh [58,p.153], Cavaretta et al [12] established that for the sequence $\{(m_i, n_i)\}_{i=1}^{\infty}$ for which there exists an α with $0 \leq \alpha < \infty$ such that

$$\lim_{i\to\infty} m_i = \infty \qquad \text{and} \qquad \lim_{i\to\infty} \frac{n_i}{m_i} = \alpha, \tag{1.8.1}$$

one has

$$\lim_{i\to\infty} \left\{ r_{m_i}^{(m_i,n_i)}(z; f) - \sum_{j=0}^{l-1} P_{m_i,n_i,j}(z; f) \right\} = 0, \quad \forall |z| < \rho^{1+l(1+\alpha)},$$

and, if $\alpha > 0$, then

$$\lim_{i\to\infty} \left\{ s_{n_i}^{(m_i,n_i)}(z^{-1}; f) - \sum_{j=1}^{l-1} Q_{m_i,n_i,j}(z^{-1}; f) \right\} = 0. \quad \forall |z| > \rho^{1-l(1+1/\alpha)}$$

T.E.Price [34] obtained an extension of this result by considering average of the polynomials .

Define the averages

$$R_{m_i}^{(m_i,n_i)}(z; f) = \frac{1}{m} \sum_{q=0}^{m-1} r_{m_i}^{(m_i,n_i)}(z; f),$$

$$S_{n_i}^{(m_i,n_i)}(z^{-1};f) = \frac{1}{m}\sum_{q=0}^{m-1} s_{n_i}^{(m_i,n_i)}(z^{-1};f),$$

$$U_{m_i,n_i,j}(z;f) = \frac{1}{m}\sum_{q=0}^{m-1} P_{m_i,n_i,j}(z;f)$$

and

$$V_{n_i,m_i,j}(z^{-1};f) = \frac{1}{m}\sum_{q=0}^{m-1} Q_{n_i,m_i,j}(z^{-1};f)$$

and if β is the least positive integer such that $\beta m > l$ then with these notations and the sequence $\{(m_i, n_i)\}$ satifying (1.8.1), Price [34] proved that

$$\lim_{i\to\infty}\left\{R_{m_i}^{(m_i,n_i)}(z;f) - \sum_{j=0}^{l-1} U_{m_i,n_i,j}(z;f)\right\} = 0, \ \forall \ |z| < \rho^{1+\beta m(1+\alpha)}$$

and for $\alpha > 0$,

$$\lim_{i\to\infty}\left\{S_{n_i}^{(m_i,n_i)}(z^{-1};f) - \sum_{j=0}^{l-1} V_{n_i,m_i,j}(z^{-1};f)\right\} = 0, \ \forall |z| > \rho^{1-\beta m(1+\frac{1}{\alpha})}.$$

1.9 Saff and Sharma [40] gave analogous extensions for overconvergence by considering rational interpolants to functions in A_ρ. They considered rational interpolants which have poles equally spaced on the circle $|z| = \sigma, \sigma > 1$. To obtain precise regions of equiconvergence for the aforesaid rational functions, they considered two schemes for extending the corresponding result of J.L.Walsh. The first scheme interpolates $f(z)$ in the roots of unity, while the second consists of best l_2 approximants to $f(z)$ on the unit circle.

In place of the Lagrange polynomial $L_{n-1}(z;f)$ they had taken the unique function $R_{n+m,n}(z;f)$ of the form

$$R_{n+m,n}(z;f) = \frac{B_{n+m,n}(z;f)}{z^n - \sigma^n}, \qquad B_{n+m,n}(z;f) \in \Pi_{n+m},$$

which interpolates f in the $(n+m+1)^{th}$ roots of unity. Since the $(n-1)th$ partial sum of f, $P_{n-1}(z;f)$ is also the least squares approximation to f from Π_{n-1} on the unit circle, Saff and Sharma [40] replaced this polynomial by the unique rational function

$$r_{n+m,n}(z;f) = \frac{P_{n+m,n}(z;f)}{z^n - \sigma^n}, \qquad P_{n+m,n}(z;f) \in \Pi_{n+m},$$

which minimizes the integral

$$\int_{|z|=1} |f(z) - r(z)|^2 |dz|$$

over all rational functions of the form $p(z)/(z^n - \sigma^n), p \in \Pi_{n+m}$. For fixed $m \geq -1$ they have shown that the sequences $\{R_{n+m,n}(z; f)\}_{n=1}^{\infty}$ and $\{r_{n+m,n}(z; f)\}_{n=1}^{\infty}$ converge to $f(z), \forall\ |z| < min(\sigma, \rho)$. Furthermore, if $\rho > \sigma$, then for all $|z| > \rho$ the two sequences converge to zero for $m = -1$ and to $\sum_{k=0}^{m} a_k z^k$ for $m \geq 0$, but the difference of two sequences converges to zero, $\forall |z| < \rho^2$ if $\sigma \geq \rho^2$ and $\forall |z| \neq \sigma$ if $\rho^2 > \sigma$. Moreover the result is sharp.

Saff and Sharma [40] also extended this result in the spirit of Theorem 1.1.1. They have shown that $f \in A_\rho$ can be written as

$$f(z) = \sum_{\nu=0}^{\infty} \left\{ \frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)} \right\}^{\nu} r_{n+m,n}(z; f, \nu),$$

where $\alpha_{n,m}(z) = 1 - z^{m+1}\sigma^{-n}$, $\beta_{n,m}(z) = z^{m+1}(z^n - \sigma^{-n})$,

$$r_{n+m,n}(z; f, \nu) = \frac{P_{n+m,n}(z; f, \nu)}{z^n - \sigma^n}, \qquad P_{n+m,n}(z; f, \nu) \in \Pi_{n+m}.$$

Set

$$\Delta_{n,m,l}^{\sigma}(z; f) = R_{n+m,n}(z; f) - \sum_{\nu=0}^{l-1} r_{n+m,n}(z; f, \nu).$$

Using above notations Saff and Sharma [40] proved that for $m \geq -1$, the sequence $\{\Delta_{n,m,l}^{\sigma}(z; f)\}_{n=1}^{\infty}$ converges to zero for $|z| < \rho^{1+l}$ if $\sigma \geq \rho^{1+l}$ and for $|z| < \sigma$ and $|z| > \sigma$ if $\sigma < \rho^{1+l}$. Moreover the result is sharp.

Motivated by the results of Totik [56], in 1988 M.A.Bokhari [5] gave some quantitative estimates for the sequence $\{\Delta_{n,m,l}^{\sigma}(z; f)\}_{n=1}^{\infty}$. For more detailed results see [3], [4], [6], [7], [8], [9], [40], [41], [48] and [49].

1.10 For extensions of Walsh's theorem on overconvergence, authors have mostly considered functions analytic inside a circle. Rivlin [39] was the first to consider functions analytic inside an ellipse in this context. Suppose $1 < \rho < \infty$. Let C_ρ be the ellipse, in z-plane, with foci at -1 and $+1$ obtained by the map $z = (w + w^{-1})/2$ from $|w| = \rho$. This mapping maps the exterior as well as the interior of $|w| = 1$ in a 1-1 conformal fashion on the (extended) z-plane with the interval $[-1, 1]$ deleted. Each pair of circles $|w| = \rho, 1/\rho$ is mapped onto the same ellipse in the z-plane, C_ρ, with foci at $(\pm 1, 0)$ and the sum of major and minor axis equal to 2ρ.

Chebyshev polynomials form an orthonormal set in an ellipse. Chebyshev polynomial of degree k is given by $T_k(z) = (w^k + w^{-k})/2$.

The mapping $z = (w + w^{-1})/2$ maps $|w| = 1$ on the interval $[-1, 1]$. Thus for $w = e^{\imath\theta}$; $0 \leq \theta \leq 2\pi$, $z = x = \frac{e^{\imath\theta} + e^{-\imath\theta}}{2} = cos\theta$. Hence for the interval $[-1, 1]$, Chebyshev polynomial of degree k reduces to

$$\begin{aligned} T_k(x) &= \frac{w^k + w^{-k}}{2} \\ &= \frac{e^{\imath k\theta} + e^{-\imath k\theta}}{2} \\ &= cos(k\theta),\ cos\theta = x,\ -1 \leq x \leq 1. \end{aligned}$$

From the trigometric identities

$$cos(n+1)\theta + cos(n-1)\theta = 2cos\theta cosn\theta$$

and

$$2cosm\theta cosn\theta = cos(n+m)\theta + cos(n-m)\theta$$

we find the relations

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

and

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{|n-m|}(x).$$

Explicit expressions for the first few Chebyshev polynomials are

$$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x, T_4(x) = 8x^4 - 8x^2 + 1.$$

Also

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_r(x) T_s(x) dx &= \pi \qquad r = s = 0 \\ &= \frac{\pi}{2} \qquad r = s \neq 0 \\ &= 0 \qquad r \neq s. \end{aligned}$$

For the detailed study of Chebyshev polynomials see [31,38]. Let $A(C_\rho)$ denote the class of functions, f, analytic inside C_ρ and having a singularity on C_ρ. Every function in $A(C_\rho)$ has Fourier-Chebyshev series expansion as

$$f(z) = {\sum_{k=0}^{\infty}}' A_k T_k(z),$$

where $T_k(z) = (w^k + w^{-k})/2$ is the Chebyshev polynomial of degree k and the stroke on the summation sign means that the first term of the sum is to be halved and

$$A_k = \frac{2}{\pi} \int_\Gamma f\left(\frac{(w+w^{-1})}{2}\right)(w^k + w^{-k})\frac{dw}{w}.$$

where Γ is $|w| = R < \rho$.

Let $q \equiv q(m,n) = mn + c$ where m is an integer , $m \geq 1$ and c is integer satisfying $0 \leq c < m$ and $0 \leq n$. Let $S_n(z; f)$ denote the n^{th} partial sum of this expansion of f in Fourier-Chebyshev series . Let $U_{n,q}(z; f)$ denote the best l_2 - approximation of degree n to $f(z)$ on $\{\xi_j^{(q)}\}_{j=1}^q$, the q zeros of $T_q(z)$. Then, Rivlin [39] proved that for $m > 1$

$$\lim_{n\to\infty} \{U_{n\,q}(z; f) - S_n(z; f)\} = 0, \qquad z \in C^o_{\rho^{2m-1}}$$

where $C^o_{\rho^{2m-1}}$ is the interior of the ellipse $C_{\rho^{2m-1}}$.

In [16] an extension of this result is derived by a different method for mixed Hermite and l_2 - approximation. As a special case it is shown that for and $lm > 1$

$$\lim_{n\to\infty} \left\{U_{n\,q}(z; f) - S_n(z; f) - \sum_{j=1}^{l-1} S_{n,j}(z; f)\right\} = 0, \qquad z \in C^o_{\rho^{2lm-1}},$$

where

$$S_{n,j}(z; f) = \sum_{k=0}^{n}(A_{2jq+k} + A_{2jq-k})T_k(z), j = 1, 2, \ldots.$$

Walsh's theorem was extended in the direction of optimal recovery [11,32], equisummability [23] and Hermie Birkhoff interpolation [13,43] as well. In fact, there are a number of papers in the direction of Walsh equiconvergence theory, and it is difficult to cite the contributions of all ; however, we may mention a few names of mathematicians, for instance Mu Le Hua [18], Bruck Rainer [35,36,37], M. Simkani [46], Z.Ziegler [45] etc. For further detailed survey see [30,44,57].

Motivated by the work of V.Totik [56], K.G.Ivanov and A.Sharma [19,20], M.P.Stojanova [50] and Lou Yuanren [28] in this dissertation we have tried to give some more extensions of Walsh overconvergence theorem. We now give chapter wise summary of the thesis.

In Chapter two an attempt is made to see how far results are valid for the average of interpolating polynomials. Here we study the pointwise behaviour of the sequence of differences of two polynomials associated with a function in A_ρ. For a special case, results

for the derivatives reproduce and generalise the few earlier results of the same chapter. Results of this chapter extend the results of T.E.Price [33], V.Totik [56], Ivanov & Sharma [19] and Lou Yuanren [26].

In Chapter three some exact results are given by considering least square approximating polynomials to generalise the Walsh's result. The behaviour of the sequence is also studied outside its region of convergence. We have succeeded in obtaining the results by considering n^{th} roots of α^n, $|\alpha| < \rho$, which generalise the results for n^{th} roots of unity given in the same chapter. Results of this chapter extend a result of A.S.Cavaretta, H.P.Dixit and A.Sharma [10] and generalise a result of M.P.Stojanova [50] and as a particular case they give results of Totik [56] and Ivanov & Sharma [20].

In Chapter four we are able to extend a few results of Chapter two by considering the averarage of the least square approximating polynomials. The results of this chapter for n^{th} roots of unity are generalised for n^{th} roots of α^n, $|\alpha| < \rho$. In a special case the last result gives a result of M.P.Stojanova [50].

Chapter five incorporates the results for Hermite interpolating polynomials. Here quantitative estimates are obtained for the growth of the derivatives of the sequence of differences of two polynomials. Results obtained here extend the results of Lou Yuanren [26] to Hermite interpolation and, in a particular case, they generalise the results of Ivanov and Sharma [20] given for Hermite interpolation.

Chapter six is devoted to the study of polynomial interpolants in z, z^{-1}. We obtain two sequences of differences of polynomials, one in z and the other in z^{-1}. Exact results for both sequences are obtained seperatly. Their behaviour outside the their region of convergence are also studied. These results extend a result of Cavaretta et al [12] for polynomials in z, z^{-1}. Further, as a special case, the results for polynomials in z give results of V.Totik [56] and K.G.Ivanov & A.Sharma [19].

Finally, in Chapter seven, we consider functions analytic in an ellipse and hence represented the by Chebyshev sereis. We obtain quantitative estimates for the sequence of differences of two polynomials which are averages of polynomials associated with interpolating polynomials to a function at the zeros and extremas of Chebyshev polynomials.

particular cases of these results give seperate results for the polynomials associated zeros and extremas of Chebyshev polynomials. Results of this chapter extend a result of T.J.Rivlin [39].

Chapter 2

WALSH OVERCONVERGENCE USING AVERAGES OF INTERPOLATING POLYNOMIALS AND THEIR DERIVATIVES

2.1 Let $\rho > 1$ and denote by A_ρ and R_ρ the set of all functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with the coefficients satisfying

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \rho^{-1} \qquad \text{and} \qquad \overline{\lim_{n\to\infty}} |a_n|^{1/n} \le \rho^{-1}$$

respectively. Put

$$P_{n-1,j}(z; f) = \sum_{k=0}^{n-1} a_{k+nj} z^k \tag{2.1.1}$$

and denote by $L_{n-1}(z; f)$ the Lagrange interpolating polynomial of degree at most $n-1$ which interpolates to f at the n^{th} roots of unity.

T.E.Price [33] extended Walsh's theorem by replacing the Lagrange interpolant on n^{th} roots of unity by the average of Lagrange interpolating polynomial. More specifically, for positive integers m and n let $\omega = e^{\frac{2\pi i}{mn}}$. Set $f_s(z) = f(z\omega^s)$ for $s = 0, \ldots, m-1$, and define the averages

$$A_{n-1,m}(z; f) = \frac{1}{m} \sum_{s=0}^{m-1} L_{n-1}(z\omega^{-s}; f_s) \tag{2.1.2}$$

and for each $j \ge 0$

$$A_{n-1,m,j}(z; f) = \frac{1}{m} \sum_{s=0}^{m-1} P_{n-1,j}(z\omega^{-s}; f_s). \tag{2.1.3}$$

Using above notations Price [33] proved

Theorem 2.1.1 [33] *Let $f \in A_\rho$ and l be a positive integer. Let β be the least positive integer such that $\beta m > l-1$. Then*

$$\lim_{n\to\infty}\left(A_{n-1,m}(z;f) - \sum_{j=0}^{l-1} A_{n-1,m,j}(z;f)\right) = 0 \qquad \forall\ |z| < \rho^{1+\beta m}, \tag{2.1.4}$$

the convergence being uniform and geometric in $|z| \le \mathcal{T} < \rho^{1+\beta m}$. Moreover, the result is best possible, in the sense that (2.1.4) fails for every z satisfying $|z| = \rho^{1+\beta m}$ for an $f \in A_\rho$.

Now let for $l \ge 1$

$$\Delta_{n-1,l}(z;f) = L_{n-1}(z;f) - \sum_{j=0}^{l-1} P_{n-1,j}(z;f)$$

and

$$K_l(|z|,\rho) = \begin{cases} \frac{|z|}{\rho^{1+l}} & \text{if} \quad |z| \ge \rho \\ \frac{1}{\rho^l} & \text{if} \quad 0 \le |z| \le \rho. \end{cases}$$

Totik [56] generalised and made exact Theorem 1.1.1, an extension of Walsh's theorem, in the following sense.

Theorem 2.1.2 [56] *If $f \in A_\rho$ then for any positive integer l and $R > 0$*

$$\lim_{n\to\infty} \max_{|z|=R} |\Delta_{n-1,l}(z;f)|^{1/n} = K_l(R,\rho).$$

Next considering the pointwise behaviour of $\Delta_{n-1,l}(z;f)$ Totik [56] proved

Theorem 2.1.3 [56] *Let $f \in A_\rho, \rho > 1$ and $l \ge 1$. Then*

(i) $$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l}(z;f) \mid^{1/n} = \frac{\mid z \mid}{\rho^{1+l}}$$

for all but at most l distinct points in $\mid z \mid > \rho$ and

(ii) $$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l}(z;f) \mid^{1/n} = \frac{1}{\rho^l}$$

for all but at most $l-1$ distinct points in $0 < \mid z \mid < \rho$.

Totik [56] also proved that the above result is best possible in the sense of

Theorem 2.1.4 [56] *Let $\rho > 1$ and $l \ge 1$.*

(i) If $z_1, \ldots, z_l$ are arbitrary l points with modulus greater than ρ then there is a

rational function $f \in A_\rho$ *with*

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l}(z_j; f) \mid^{1/n} < \frac{\mid z_j \mid}{\rho^{1+l}}, \qquad j = 1, \ldots, l.$$

(ii) If $z_1, \ldots, z_{l-1}$ *are arbitrary* $l-1$ *points in the ring* $0 < |z| < \rho$ *then there is a rational function* $f \in A_\rho$ *with*

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l}(z_j; f) \mid^{1/n} = \frac{1}{\rho^l}, \qquad j = 1, \ldots, l-1.$$

If we set

$$B_l(z; f) = \overline{\lim_{n\to\infty}} |\Delta_{n-1,l}(z_j; f)|^{1/n}, \tag{2.1.5}$$

then from the definition of $K_l(|z|, \rho)$ it follows that $B_l(z; f) \leq K_l(|z|, \rho)$. Define a set Z of points to be (l, ρ) distinguished if there is an $f \in A_\rho$ such that $B_{l,m}(z_j; f) < K_l(|z_j|, \rho)$, for each $z_j \in Z$. Suppose $Z = \{z_j\}_{j=1}^q$ is given in which $|z_j| < \rho \quad (j = 1, \ldots, \mu)$ and $|z_j| > \rho \quad (j = \mu + 1, \ldots, q)$. Ivanov and Sharma [19] find a criterion to determine whether Z is (l, ρ) distinguished or not. Set the matrix X and Y as

$$X = (z_j^i)_{1 \leq j \leq \mu,\, 0 \leq i \leq l-1}, \quad Y = (z_j^i)_{\mu+1 \leq j \leq q,\, 0 \leq i \leq l}.$$

The matrices X and Y are of order $\mu \times l$ and $(s - \mu) \times (l + 1)$ respectively. Further $M = M(X, Y)$ is same as in 1.2.2, where X is repeated $l + 1$ times and Y repeated l times and Y's begin below the last row of last X. The matrix M is of order $(ql + \mu) \times l(l + 1)$. Using these notations K.G.Ivanov and A.Sharma [19] proved

Theorem 2.1.5 [19] *Suppose* $Z = \{z_j\}_{j=1}^q$ *is a set of points in* C *such that* $|z_j| < \rho (j = 1, \ldots, \mu)$ *and* $|z_j| > \rho (j = \mu+1, \ldots, q)$. *Then the set* Z *is* (l, ρ) *distinguished iff*

$$rank M < l(l + 1).$$

Lou [28] gave some quantitative estimates for $\overline{\lim}_{n\to\infty} \max_{|z|=R} |\Delta_{n-1,l}^{(r)}(z; f)|^{1/n}$, where $\Delta_{n-1,l}^{(r)}(z; f)$ is the r^{th} derivative of $\Delta_{n-1,l}(z; f)$.

Theorem 2.1.6 [28] *For each* $f \in R_\rho (\rho > 1)$, *any integers* $l \geq 1$ *and* $r \geq 0$, *and any* $R > 0$, *there holds*

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{n-1,l}^{(r)}(z; f)|^{1/n} \leq K_l(R, \rho) \tag{2.1.6}$$

Equality holds in (2.1.6) iff $f \in A_\rho$.

Next by introducing the concept of distinguished point of degree r , Lou [28] investigated some relations between the order of pointwise convergence (or divergence) of $\Delta^{(r)}_{n-1,l}(z;f)$ and the properties of $f(z)$.

For any integer $r \geq 0$, set

$$B_l^r(z;f) := \overline{\lim_{n\to\infty}} |\Delta^{(r)}_{n-1,l}(z;f)|^{1/n}.$$

We say that η is an (l,ρ)-distinguished point of $f \in A_\rho$ of degree r if

$$B_l^\nu(\eta;f) < K_l(|\eta|,\rho), \qquad \forall\ \nu = 0,1,\ldots,r-1,$$

and consider it as r points coincided at η.

Hereafter let $\{\eta_\nu\}_{\nu=1}^s$ be a set of s points in C and p_ν denote the number of appearence of η_ν in $\{\eta_j\}_{j=1}^\nu$. Then

Theorem 2.1.7 [28] *If* $f \in R_\rho(\rho > 1), l$ *is any positive integer, and there are* $l+1$ *points* $\{\eta_\nu\}_{\nu=1}^{l+1}$ *in* $|z| > \rho$ *(or* l *points* $\{\eta_\nu\}_{\nu=1}^{l}$ *in* $|z| < \rho$*) for which*

$$B_l^{p_\nu-1}(\eta_\nu;f) < K_l(|\eta_\nu|,\rho), \qquad \nu = 1,\ldots,l+1(or\ l),$$

then $f \in R_\rho \backslash A_\rho$.

Theorem 2.1.8 [28] *Let* $f \in A_\rho(\rho > 1), l$ *be any positive integer and* $\{\eta_\nu\}_{\nu=1}^s$ *be any* s *points in* $|z| > \rho, s \leq l$, *(or in* $|z| < \rho, s \leq l-1$*), with the numbers* p_ν *of the appearence of* η_ν *in* $\{\eta_j\}_{j=1}^\nu$. *Then the neccessary and sufficient condition for*

$$B_l^{p_\nu-1}(\eta_\nu;f) < K_l(|\eta_\nu|,\rho), \qquad \nu = 1,\ldots,s$$

is

$$f(z) = w_s(z)G_s(z) + G_0(z)$$

where $w_s(z) := \prod_{j=1}^s (z-\eta_j)$, $G_0(z) \in R_\rho/A_\rho$ *and* $G_s(z) = \sum_{j=0}^\infty \alpha_j z^j \in A_\rho$ *with*

$$\alpha_{(l+1)n-\nu} = 0 \ \ (\text{or}\ \ \alpha_{ln-\nu} = 0), \qquad \nu = 1,2,\ldots,s.$$

Generalising a Theorem of Ivanov and Sharma [19] for the case that the points of $\{z_j\}_1^s$ can be coincided with each other, let $Z = \{\eta_j\}_{j=1}^s$ with $|\eta_j| < \rho, j = 1,\ldots,\mu$ and

$|\eta_j| > \rho, j = \mu + 1, \ldots, s$, and p_ν denote the number of appearence of η_ν in $\{\eta_j\}_{j=1}^{\nu}, \nu = 1, \ldots, s$. Calling a set Z an (l, ρ)-distinguished set if there exists an $f \in A_\rho$ such that $B_l^{p_\nu - 1}(\eta_\nu; f) < K_l(|\eta_\nu|, \rho), \nu = 1, \ldots, s$, define matrices X and Y as follows

$$X = [S_{i,j}]_{\mu \times l}, \qquad Y = [\hat{S}_{i,j}]_{(s-\mu)\times(l+1)},$$

where the elements are given by

$$(z^j)^{(p_i - 1)}|_{z=\eta_i} = \begin{cases} S_{i,j}, & \text{if } i = 1, \ldots, \mu; \quad j = 0, 1, \ldots, l; \\ \hat{S}_{i-\mu,j}, & \text{if } i = \mu + 1, \ldots, s; \quad j = 0, 1, \ldots, l + 1, \end{cases}$$

and define $M = M(X, Y)$ same as in 1.2.2. Then

Theorem 2.1.9 [28] *Suppose $Z = \{\eta_j\}_{j=1}^{s}$ is a set of points in C with the repeated number p_j of η_j in $\{\eta_\nu\}_{\nu=1}^{j}$ such that $|\eta_j| < \rho (j = 1, \ldots, \mu)$ and $|\eta_j| > \rho (j = \mu + 1, \ldots, s)$. Then the set Z is (l, ρ)-distinguished iff*

$$rank\ M < l(l + 1).$$

Motivated by the above results of Totik [56], Ivanov & Sharma [19] and Lou Yuanren [28], in this chapter we give some exact results and quantitative estimates for $\overline{\lim}_{n\to\infty} \max_{|z|=R} |A_{n-1,m}(z; f) - \sum_{j=0}^{l-1} A_{n-1,m,j}(z; f)|^{1/n}$, for points in $|z| \leq \rho$ and points in $|z| > \rho$ seperatly. Further, distinguished set is considerd containing the points in $|z| < \rho$ and $|z| > \rho$ simulatneosly. We also consider the pointwise behaviour of the sequence $\{A_{n-1,m}(z; f) - \sum_{j=0}^{l-1} A_{n-1,m,j}(z; f)\}$ from which we are able to state a result for its behaviour outside its region of convergence. We further generalise these results by considering derivative of the above sequence. The results of this chapter, as a particular case give all the above stated theorems.

2.2 Let for $l \geq 1$

$$\Delta_{n-1,l,m}(z; f) = A_{n-1,m}(z; f) - \sum_{j=0}^{l-1} A_{n-1,m,j}(z; f)$$

where $A_{n-1,m}(z; f)$ and $A_{n-1,m,j}(z; f)$ are given by (2.1.2) and (2.1.3) respectively. Further,let

$$\begin{aligned} K_{\beta,m}(R, \rho) &= \frac{R}{\rho^{1+\beta m}}, \quad \text{if} \quad R \geq \rho \\ &= \frac{1}{\rho^{\beta m}} \quad \text{if} \quad 0 \leq R \leq \rho. \end{aligned} \tag{2.2.1}$$

then Theorem 2.1.1 give for $R > \rho$

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{n-1,l,m}(z;f)|^{1/n} \le K_{\beta,m}(R,\rho).$$

In the present section we show that for $R > \rho$ equality always hold. We extend this result to the case $R \le \rho$ also. It is easy to see that (see e.g. [33])

$$A_{n-1,m,j} = \begin{cases} P_{n-1,j} & \text{if } j = pm,\ p \ge 0, \text{an integer} \\ 0 & \text{otherwise} \end{cases}$$

where $P_{n-1,j}(z;f)$ is given by (2.1.1). Hence, if β is the least positive integer such that $\beta m > l-1$, then

$$\begin{aligned} \Delta_{n-1,l,m}(z;f) &= A_{n-1,m}(z;f) - \sum_{j=0}^{l-1} A_{n-1,m,j}(z;f) \\ &= A_{n-1,m}(z;f) - \sum_{j=0}^{\beta-1} A_{n-1,m,jm}(z;f) \\ &= A_{n-1,m}(z;f) - \sum_{j=0}^{\beta-1} P_{n-1,m,jm}(z;f). \end{aligned} \tag{2.2.2}$$

Also from [33]

$$A_{n-1,m}(z;f) = \sum_{j=0}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn} z^k.$$

Hence by (2.2.2) and the definition of $P_{n-1,j}(z;f)$ and β we have

$$\begin{aligned} \Delta_{n-1,l,m}(z;f) &= \sum_{j=0}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn} z^k - \sum_{j=0}^{\beta-1}\sum_{k=0}^{n-1} a_{k+jmn} z^k \\ &= \sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn} z^k. \end{aligned} \tag{2.2.3}$$

If we set

$$g_{\beta,m}(R) = \overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{n-1,l,m}(z;f)|^{1/n},$$

then we have

Theorem 2.2.1 *If $f \in A_\rho$, l is a positive integer and β is the least positive integer such that $\beta m > l-1$ and $R > 0$ then*

$$g_{\beta,m}(R) = K_{\beta,m}(R,\rho)$$

Proof : Since $f \in A_\rho$, we have

$$a_k = \mathcal{O}(\rho - \epsilon)^{-k} \tag{2.2.4}$$

for every ϵ satisfying $0 < \epsilon < \rho - 1$ and $k \geq k_0(\epsilon)$. Let R be fixed, $|z| = R$ and if $R < \rho$ then we assume $\epsilon > 0$ so small that $R < \rho - \epsilon$ be satisfied as well. Then by (2.2.3) we obtain

$$\begin{aligned}
\Delta_{n-1,l,m}(z; f) &= \sum_{j=\beta}^{\infty} \sum_{k=0}^{n-1} a_{k+jmn} z^k \\
&= \mathcal{O}\left(\sum_{j=\beta}^{\infty} \sum_{k=0}^{n-1} \frac{|z|^k}{(\rho - \epsilon)^{k+jmn}} \right) \\
&= \mathcal{O} \begin{cases} \frac{R^n}{(\rho-\epsilon)^{(1+\beta m)n}}, & \text{if} \quad R \geq \rho \\ \frac{1}{(\rho-\epsilon)^{\beta mn}} & \text{if} \quad 0 < R \leq \rho \end{cases}
\end{aligned}$$

Thus on taking n^{th} roots we have

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{n-1,l,m}(z; f)|^{1/n} \leq K_{\beta,m}(R, \rho - \epsilon),$$

ϵ being arbitraily small we have

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{n-1,l,m}(z; f)|^{1/n} \leq K_{\beta,m}(R, \rho).$$

To prove the opposite inequality let first $R \geq \rho$, then

$$\begin{aligned}
\Delta_{n-1,l,m}(z; f) &= \sum_{j=\beta}^{\infty} \sum_{k=0}^{n-1} a_{k+jmn} z^k \\
&= \sum_{k=0}^{n-\beta m-2} a_{k+\beta mn} z^k + \sum_{k=n-\beta m-1}^{n-1} a_{k+\beta mn} z^k + \\
&\quad + \sum_{j=\beta+1}^{\infty} \sum_{k=0}^{n-1} a_{k+jmn} z^k.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=n-\beta m-1}^{n-1} a_{k+\beta mn} z^k &= \Delta_{n-1,l,m}(z; f) - \sum_{k=0}^{n-\beta m-2} a_{k+\beta mn} z^k - \\
&\quad - \sum_{j=\beta+1}^{\infty} \sum_{k=0}^{n-1} a_{k+jmn} z^k
\end{aligned}$$

gives, by Cauchy integral formula, for $n - \beta m - 1 \leq k \leq n - 1$,

$$\begin{aligned}
a_{k+\beta mn} &= \frac{1}{2\pi i} \int_{|z|=R} \frac{\Delta_{n-1,l,m}(z; f)}{z^{k+1}} dz - \frac{1}{2\pi i} \sum_{k'=0}^{n-\beta m-2} a_{k'+\beta mn} \int_{|z|=R} \frac{z^{k'}}{z^{k+1}} dz \\
&\quad - \frac{1}{2\pi i} \int_{|z|=R} \frac{\sum_{j=\beta+1}^{\infty} \sum_{k'=0}^{n-1} a_{k'+jmn} z^{k'}}{z^{k+1}} dz.
\end{aligned}$$

Since $\int_{|z|=R} \frac{z^{k'}}{z^{k+1}} dz$ is non zero only for $k = k'$, the middle integral on the right hand side in above equation is zero. Then by the definition of $g_{\beta,m}(R)$ and (2.2.4) for every $n \geq n_0(\epsilon)$ and a constant M, which need not be same at each occurrence we have

$$\begin{aligned} | a_{k+\beta mn} | &\leq M \frac{(g_{\beta,m}(R)+\epsilon)^n}{R^k} + \mathcal{O}\left(\frac{R^n}{R^k(\rho-\epsilon)^{n+(\beta+1)mn}}\right) \\ &\leq M \frac{(g_{\beta,m}(R)+\epsilon)^n}{R^k} + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{n(1+(\beta+1)m)}}\right). \end{aligned}$$

Let $\epsilon > 0$ be so small that

$$(\rho-\epsilon)^{-(1+(\beta+1)m)} < \rho^{-(1+\beta m)}.$$

Thus,

$$(g_{\beta,m}(R)+\epsilon)^n \geq \frac{R^k}{M}\left(| a_{k+\beta mn} | - \mathcal{O}\left(\frac{1}{\rho^{n(1+\beta m)}}\right)\right)$$

hence,

$$g_{\beta,m}(R)+\epsilon \geq \varlimsup_{n\to\infty} \left\{|a_{k+\beta mn}|^{\frac{1}{k+\beta mn}}\right\}^{\frac{k+\beta mn}{n}} \left\{\frac{R^k}{M}\right\}^{\frac{1}{n}}.$$

Now since $n - \beta m - 1 \leq k \leq n-1$ we have, $\lim_{n\to\infty} \frac{k}{n} = 1$ and so

$$g_{\beta,m}(R)+\epsilon \geq \frac{R}{\rho^{1+\beta m}}.$$

Since ϵ is arbitrary, this yeilds

$$g_{\beta,m}(R) \geq \frac{R}{\rho^{1+\beta m}} \qquad \text{for} \qquad R \geq \rho.$$

For the case $0 < R \leq \rho$, we write

$$\begin{aligned} \Delta_{n-1,l,m}(z;f) &= \sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k \\ &= \sum_{k=0}^{\beta m-1} a_{k+\beta mn}z^k + \sum_{k=\beta m}^{n-1} a_{k+\beta mn}z^k + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k \end{aligned}$$

whence,

$$\sum_{k=0}^{\beta m-1} a_{k+\beta mn}z^k = \Delta_{n-1,l,m}(z;f) - \sum_{k=\beta m}^{n-1} a_{k+\beta mn}z^k - \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k.$$

By Cauchy integral formula we have, for $0 \leq k \leq \beta m - 1$,

$$\begin{aligned} a_{k+\beta mn} &= \frac{1}{2\pi i}\int_{|z|=R} \frac{\Delta_{n-1,l,m}(z;f)}{z^{k+1}}dz - \frac{1}{2\pi i}\sum_{k'=\beta m}^{n-1} a_{k'+\beta mn}\int_{|z|=R}\frac{z^{k'}}{z^{k+1}}dz \\ &\quad -\frac{1}{2\pi i}\int_{|z|=R} \frac{\sum_{j=\beta+1}^{\infty}\sum_{k'=0}^{n-1} a_{k'+jmn}z^{k'}}{z^{k+1}}dz. \end{aligned}$$

Using the same arguments as earlier, we then have,

$$
\begin{aligned}
\mid a_{k+\beta mn} \mid &\leq M\frac{(g_{\beta,m}(R)+\epsilon)^n}{R^k}+\mathcal{O}\left(\frac{1}{R^k(\rho-\epsilon)^{(\beta+1)mn}}\right) \\
&\leq M(g_{\beta,m}(R)+\epsilon)^n+\mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(\beta+1)mn}}\right).
\end{aligned}
$$

Let $\epsilon > 0$ be so small that

$$(\rho-\epsilon)^{-(\beta+1)} < \rho^{-\beta}$$

then,

$$(g_{\beta,m}(R)+\epsilon)^n \geq \frac{1}{M}\left(|a_{k+\beta mn}|-\mathcal{O}\left(\frac{1}{\rho^{\beta mn}}\right)\right)$$

or,

$$
\begin{aligned}
g_{\beta,m}(R)+\epsilon &\geq \varlimsup_{n\to\infty}\left\{|a_{k+\beta mn}|^{\frac{1}{k+\beta mn}}\right\}^{\frac{k+\beta mn}{n}}\left(\frac{1}{M}\right)^{\frac{1}{n}} \\
&= \frac{1}{\rho^{\beta m}}.
\end{aligned}
$$

Since ϵ is arbitrary

$$g_{\beta,m}(R) \geq \frac{1}{\rho^{\beta m}} \qquad \text{for} \qquad 0 < R \leq \rho$$

which completes the proof.

Note that for $R = 0$ that is $z = 0$

$$
\begin{aligned}
\Delta_{n-1,l,m}(z;f) &= \sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k \\
&= \sum_{j=\beta}^{\infty} a_{jmn}.
\end{aligned}
$$

For

$$F(z) = \frac{1}{1-(z/\rho)^{(\beta+1)m}}$$

$a_{\beta mn} = 0, \ \forall n$. Hence

$$
\begin{aligned}
\Delta_{n-1,l,m}(0;F) &= \sum_{j=\beta+1}^{\infty} a_{jmn} \\
&= \mathcal{O}\left(\rho^{-(\beta+1)mn}\right),
\end{aligned}
$$

whence

$$\varlimsup_{n\to\infty}\max_{|z|=R=0}|\Delta_{n-1,l,m}(z;F)|^{1/n} \leq \rho^{-(\beta+1)mn} < \frac{1}{\rho^{\beta mn}}.$$

Thus,

Remark 2.2.1 For $R = 0$ Theorem 2.2.1 does not hold.

Remark 2.2.2 For $m = 1$ Theorem 2.2.1 reduces to Theorem 2.1.2.

Corollary 2.2.1 *If $l \geq 1$, f is analytic in an open domain containing $|z| \leq 1$ and $g_{\beta,m}(R) = K_{\beta,m}(R,\rho)$ for some $R > 0, \rho > 1$ then $f \in A_\rho$.*

Proof Given that f is analytic in an open domain containing $|z| \leq 1$. Hence $f \in A_{\rho'}$ for some $\rho' > 1$. Thus by Theorem 2.2.1 $g_{\beta,m}(R) = K_{\beta,m}(R,\rho')$, and from the hypothesis $g_{\beta,m}(R) = K_{\beta,m}(R,\rho)$. That is $K_{\beta,m}(R,\rho') = K_{\beta,m}(R,\rho)$ and hence $\rho' = \rho$ which gives $f \in A_\rho$.

2.3 In this section by defining the concept of a distinguished set we study the properties of a set cointaining the points in $|z| > \rho$ and $|z| < \rho$ simultaneously, regarding the number of points in both the regions.

If we set

$$B_{l,m}(z;f) = \overline{\lim_{n\to\infty}} |\Delta_{n-1,l,m}(z;f)|^{1/n}, \tag{2.3.1}$$

then from the definition of $K_{\beta,m}(|z|,\rho)$ and Theorem 2.2.1 it follows that $B_{l,m}(z;f) \leq K_{\beta,m}(|z|,\rho)$. Define a set Z of points to be (β,m,ρ) distinguished if there is an $f \in A_\rho$ such that $B_{l,m}(z_j;f) < K_{\beta,m}(|z_j|,\rho)$, for each $z_j \in Z$. Suppose $Z = \{z_j\}_{j=1}^{q}$ is given in which $|z_j| < \rho \quad (j = 1,\ldots,\mu)$ and $|z_j| > \rho \quad (j = \mu+1,\ldots,q)$. We want to find a criterion to determine whether Z is (β,m,ρ) distinguished or not. Set the matrix X and Y as

$$X = \begin{pmatrix} 1 & z_1 & \cdots & z_1^{\beta m-1} \\ 1 & z_2 & \cdots & z_2^{\beta m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_\mu & \cdots & z_\mu^{\beta m-1} \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & z_{\mu+1} & \cdots & z_{\mu+1}^{\beta m} \\ 1 & z_{\mu+2} & \cdots & z_{\mu+2}^{\beta m} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_q & \cdots & z_q^{\beta m} \end{pmatrix}$$

The matrices X and Y are of order $\mu \times \beta m$ and $(s-\mu) \times (\beta m + 1)$ respectively. Define

$$M(X,Y) = \begin{pmatrix} X & & & \\ & X & & 0 \\ & & \ddots & \\ & 0 & & X \\ Y & & & \\ & Y & & 0 \\ & & \ddots & \\ & 0 & & Y \end{pmatrix},$$

where X is repeated $\beta m + 1$ times and Y repeated βm times and Y's begin below the last row of the last X. The matrix M is of order $(q\beta m + \mu) \times \beta m(\beta m + 1)$. We now formulate

Theorem 2.3.1 *Suppose $Z = \{z_j\}_{j=1}^{q}$ is a set of points in C such that $|z_j| < \rho (j = 1, \ldots, \mu)$ and $|z_j| > \rho (j = \mu + 1, \ldots, q)$. Then the set Z is (β, m, ρ) distinguished iff*

$$rank M < \beta m(\beta m + 1).$$

Proof : First suppose $rank M < \beta m(\beta m+1)$. Then there exists a non-zero vector $b = (b_0, b_1, \ldots, b_{\beta m(\beta m+1)-1})$ such that

$$M.b^T = 0. \tag{2.3.2}$$

Set

$$\begin{aligned} f(z) &= \sum_{N=0}^{\infty} a_N z^N \\ &= \left\{b_0 + b_1 z + \ldots + b_{\beta m(\beta m+1)-1} z^{\beta m(\beta m+1)-1}\right\} \left\{1 - \left(\frac{z}{\rho}\right)^{\beta m(\beta m+1)}\right\}^{-1}. \end{aligned}$$

Clearly $f \in A_\rho$ and that

$$a_N = b_k \rho^{-\beta m(\beta m+1)\nu} \tag{2.3.3}$$

where $N = \beta m(\beta m+1)\nu + k$, $k = 0, 1, \ldots, \beta m(\beta m+1) - 1$ and $\nu \geq 0$. From (2.3.2) and (2.3.3), we have

$$\sum_{k=0}^{\beta m-1} a_{\beta mn+k} z_j^k = 0 \qquad \text{for each } n \geq 0 \text{ and } j = 1, 2, \ldots, \mu \tag{2.3.4}$$

and

$$\sum_{k=0}^{\beta m} a_{(\beta m+1)n+k} z_j^k = 0 \qquad \text{for each } n \geq 0 \text{ and } j = \mu+1, \ldots, q. \tag{2.3.5}$$

For any integer $n > 0$ let w and t be determined by

$$\beta mn + t = (\beta m + 1)w, \qquad 0 \leq t < \beta m + 1$$

then for $j \geq \mu + 1$ from (2.3.5) we have

$$\begin{aligned}
\sum_{k=0}^{n-1} a_{k+\beta mn} z_j^k &= \sum_{k=0}^{t-1} a_{k+\beta mn} z_j^k + \sum_{k=t}^{n-1} a_{k+\beta mn} z_j^k \\
&= \sum_{k=0}^{t-1} a_{k+\beta mn} z_j^k + \left(a_{t+\beta mn} z_j^t + a_{t+1+\beta mn} z_j^{t+1} + \ldots + a_{t+\beta m+\beta mn} z_j^{t+\beta m}\right) \\
&\quad + \ldots + \left(a_{n-1-\beta m+\beta mn} z_j^{n-1-\beta m} + a_{n-1-\beta m+1+\beta mn} z_j^{n-1-\beta m+1}\right. \\
&\quad \left. + \ldots + a_{n-1-\beta m+\beta m+\beta mn} z_j^{n-1-\beta m+\beta m}\right) \\
&= \sum_{k=0}^{t-1} a_{k+\beta mn} z_j^k + \left(a_{(\beta m+1)w} z_j^{(\beta m+1)w-\beta mn} + \right. \\
&\quad \left. a_{(\beta m+1)w+1} z_j^{(\beta m+1)w-\beta mn+1} + \ldots + a_{(\beta m+1)w+\beta m} z_j^{(\beta m+1)w-\beta mn+\beta m}\right) \\
&\quad + \ldots + \left(a_{(n-1)(\beta m+1)} z_j^{(n-1)(\beta m+1)-\beta mn} + \right. \\
&\quad a_{(n-1)(\beta m+1)+1} z_j^{(n-1)(\beta m+1)-\beta mn+1} + \ldots \\
&\quad \left. + a_{(n-1)(\beta m+1)+\beta m} z_j^{(n-1)(\beta m+1)-\beta mn+\beta m}\right) \\
&= \sum_{k=0}^{t-1} a_{k+\beta mn} z_j^k + \sum_{k=0}^{\beta m} \sum_{p=w}^{n-1} a_{(\beta m+1)p+k} z_j^{(\beta m+1)p+k-mln} \\
&= \sum_{k=0}^{t-1} a_{k+\beta mn} z_j^k + 0 \qquad (\text{from } (2.3.5)) \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{\beta mn}}\right).
\end{aligned} \tag{2.3.6}$$

Thus for $\mu < j \leq q$ we obtain

$$\begin{aligned}
\Delta_{n-1,l,m}(z_j; f) &= \sum_{i=\beta}^{\infty} \sum_{k=0}^{n-1} a_{k+imn} z_j^k \\
&= \sum_{k=0}^{n-1} a_{k+\beta mn} z_j^k + \sum_{i=\beta+1}^{\infty} \sum_{k=0}^{n-1} a_{k+imn} z_j^k \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{\beta mn}}\right) + \mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{n(m(\beta+1)+1)}}\right).
\end{aligned} \tag{2.3.7}$$

Now we claim that for $|z| > \rho$, by choosing ϵ sufficiently small, we can find η, a positive number, such that

$$\frac{1}{(\rho-\epsilon)^{n\beta m}} \leq \left(\frac{|z|}{\rho^{\beta m+1}} - \eta\right)^n \tag{2.3.8}$$

and

$$\frac{|z|^n}{(\rho-\epsilon)^{(1+(\beta+1)m)n}} \leq \left(\frac{|z|}{\rho^{\beta m+1}} - \eta\right)^n. \tag{2.3.9}$$

To see this let $\epsilon_1 > 0$ be so small that

$$\rho^{\beta m} < k(\rho-\epsilon_1)^{\beta m}, \qquad \text{where} \qquad k = \frac{|z|}{\rho} > 1 \tag{2.3.10}$$

and ϵ_2 be such that

$$\rho^{(\beta m+1)} < (\rho-\epsilon_2)^{1+(\beta+1)m} \tag{2.3.11}$$

and consider

$$\epsilon = min(\epsilon_1, \epsilon_2). \tag{2.3.12}$$

Then from (2.3.10) and (2.3.12)

$$\rho^{\beta m} < \frac{|z|}{\rho}(\rho-\epsilon)^{\beta m}$$

which gives

$$k_1 := \frac{|z|}{\rho^{\beta m+1}} - \frac{1}{(\rho-\epsilon)^{\beta m}} > 0. \tag{2.3.13}$$

Similarly from (2.3.11) and (2.3.12)

$$\rho^{\beta m+1} < (\rho-\epsilon)^{1+(\beta+1)m} \tag{2.3.14}$$

$$k_2 := \frac{|z|}{\rho^{\beta m+1}} - \frac{|z|}{(\rho-\epsilon)^{1+(\beta+1)m}} > 0. \tag{2.3.15}$$

Now choose η such that

$$0 < \eta < min(k_1, k_2) \tag{2.3.16}$$

this together with (2.3.13) gives that

$$0 < \eta \leq \frac{|z|}{\rho^{\beta m+1}} - \frac{1}{(\rho-\epsilon)^{\beta m}}$$

hence

$$\frac{1}{(\rho-\epsilon)^{\beta m}} < \frac{|z|}{\rho^{\beta m+1}} - \eta$$

or,

$$\frac{1}{(\rho-\epsilon)^{n\beta m}} < \left(\frac{|z|}{\rho^{\beta m+1}} - \eta\right)^n. \tag{2.3.17}$$

Similarly from (2.3.15) and (2.3.16) we have

$$0 < \eta \leq \frac{|z|}{\rho^{\beta m+1}} - \frac{|z|}{(\rho-\epsilon)^{1+(\beta+1)m}}$$

$$\frac{|z|}{(\rho-\epsilon)^{1+(\beta+1)m}} < \frac{|z|}{\rho^{\beta m+1}} - \eta$$

and hence

$$\frac{|z|^n}{(\rho-\epsilon)^{n(1+(\beta+1)m)}} < \left(\frac{|z|}{\rho^{\beta m+1}} - \eta\right)^n \tag{2.3.18}$$

this together with (2.3.17) gives the result. Hence from (2.3.7), (2.3.8) and (2.3.9) for $j \geq \mu + 1$ we have

$$\Delta_{n-1,l,m}(z_j; f) = \mathcal{O}\left(\frac{|z_j|}{(\rho-\epsilon)^{\beta m+1}} - \eta\right)^n. \tag{2.3.19}$$

Here and elsewhere ϵ and η will denote sufficiently small positive numbers which are not same at each occurence. Now, let for any positive integer n, w and t be determined by

$$\beta m w + t = (\beta m + 1)n, \qquad 0 \leq t < \beta m.$$

Then for $0 \leq j \leq \mu$ from (2.3.4)

$$\begin{aligned}
\sum_{k=0}^{n-1} a_{k+\beta mn} z_j^k &= \sum_{k=\beta mn}^{\beta mn+n-1} a_k z_j^{k-\beta mn} \\
&= \sum_{k=\beta mn}^{\beta mw-1} a_k z_j^{k-\beta mn} + \sum_{k=\beta mw}^{(\beta m+1)n-1} a_k z_j^{k-\beta mn} \\
&= \sum_{i=n}^{w-1} \sum_{k=0}^{\beta m-1} a_{k+\beta mi} z_j^{k+\beta m(i-n)} + \sum_{k=\beta mw}^{(\beta m+1)n-1} a_k z_j^{k-\beta mn} \\
&= 0 + \sum_{k=0}^{t-1} a_{k+\beta mw} z_j^{k+\beta m(w-n)} \qquad \text{(from (2.3.4))} \\
&= \mathcal{O}\left(\frac{|z_j|^{\beta m(w-n)}}{(\rho-\epsilon)^{\beta mw}}\right) \\
&= \mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{(\beta m+1)n}}\right)
\end{aligned}$$

whence

$$\begin{aligned}
\Delta_{n-1,l,m}(z_j; f) &= \sum_{k=0}^{n-1} a_{k+\beta mn} z_j^k + \sum_{i=\beta+1}^{\infty} \sum_{k=0}^{n-1} a_{k+imn} z_j^k \\
&= \mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{(\beta m+1)n}} + \frac{1}{(\rho-\epsilon)^{(\beta+1)mn}}\right).
\end{aligned} \tag{2.3.20}$$

Proceeding as before we can show that for $|z| < \rho$ by choosing ϵ sufficiently small we can find η, a positive number, such that

$$\frac{|z|^n}{(\rho-\epsilon)^{(\beta m+1)n}} \leq \left(\frac{1}{\rho^{\beta m}} - \eta\right)^n \tag{2.3.21}$$

and

$$\frac{1}{(\rho-\epsilon)^{n(\beta+1)m}} \leq \left(\frac{1}{\rho^{\beta m}}-\eta\right)^{n}. \qquad (2.3.22)$$

Hence from (2.3.20), (2.3.21) and (2.3.22) for $0 \leq j \leq \mu$ we have

$$\Delta_{n-1,l,m}(z_j;f) = \mathcal{O}\left(\frac{1}{\rho^{\beta m}}-\eta\right)^{n} \qquad \text{for } 0 \leq j \leq \mu.$$

This together with (2.3.19) gives

$$B_{l,m}(z_j;f) < K_{\beta,m}(|z_j|,\rho), \;\; j = 0,\ldots,q.$$

For the converse part suppose $B_{l,m}(z_j;f) < K_{\beta,m}(|z_j|,\rho)$ $(j = 1,2,\ldots,q)$ for some $f \in A_\rho$ and that $rankM = \beta m(\beta m+1)$. Set

$$h(z) = \Delta_{n-1,l,m}(z;f) - z^{\beta m}\Delta_{n,l,m}(z;f).$$

Hence

$$\begin{aligned}
h(z) &= \Delta_{n-1,l,m}(z;f) - z^{\beta m}\Delta_{n,l,m}(z;f)\\
&= \sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k - z^{\beta m}\sum_{j=\beta}^{\infty}\sum_{k=0}^{n} a_{k+jm(n+1))_j}z^k\\
&= \sum_{k=0}^{n-1} a_{k+\beta mn}z^k + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k -\\
&\quad -z^{\beta m}\sum_{k=0}^{n} a_{k+\beta m(n+1)}z^k - z^{\beta m}\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n} a_{k+jm(n+1)}z^k\\
&= \sum_{k=0}^{n-1} a_{k+\beta mn}z^k - \sum_{k=\beta m}^{n+\beta m} a_{k+\beta mn}z^k +\\
&\quad + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k - z^{\beta m}\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n} a_{k+jm(n+1)}z^k\\
&= \sum_{k=0}^{\beta m-1} a_{k+\beta mn}z^k + \sum_{k=\beta m}^{n-1} a_{k+\beta mn}z^k - \sum_{k=\beta m}^{n-1} a_{k+\beta mn}z^k -\\
&\quad \sum_{k=n}^{n+\beta m} a_{k+\beta mn}z^k + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k - z^{\beta m}\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n} a_{k+jm(n+1)}z^k\\
&= \sum_{k=0}^{\beta m-1} a_{k+\beta mn}z^k - \sum_{k=0}^{\beta m} a_{k+(1+\beta m)n}z^{k+n} +\\
&\quad + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1} a_{k+jmn}z^k - z^{\beta m}\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n} a_{k+jm(n+1)}z^k\\
&= \sum_{k=0}^{\beta m-1} a_{k+\beta mn}z^k - \sum_{k=0}^{\beta m} a_{k+(1+\beta m)n}z^{k+n} + \mathcal{O}((K_{\beta+1,m}(|z_j|,\rho-\epsilon))^n). \qquad (2.3.23)
\end{aligned}$$

Hence for $0 \leq j \leq \mu$ from (2.3.21) and (2.3.22) we have

$$\begin{aligned} h(z_j) &= \sum_{k=0}^{\beta m-1} a_{k+\beta mn} z_j^k + \mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{(\beta m+1)n}} + \frac{1}{(\rho-\epsilon)^{(\beta+1)mn}}\right) \\ &= \sum_{k=0}^{\beta m-1} a_{k+\beta mn} z_j^k + \mathcal{O}\left(\frac{1}{\rho^{\beta m}} - \eta\right)^n . \end{aligned} \tag{2.3.24}$$

Since from hypothesis $B_{l,m}(z_j; f) < K_{\beta,m}(|z_j|, \rho)$ $(j = 1, 2, \ldots, \mu)$, i.e.,

$$\overline{\lim_{n\to\infty}} |\Delta_{n-1,l,m}(z_j; f)|^{1/n} < \frac{1}{\rho^{\beta m}}.$$

We can put

$$\overline{\lim_{n\to\infty}} |\Delta_{n-1,l,m}(z_j; f)|^{1/n} = \frac{1}{\rho^{\beta m}} - \eta, \;\; \eta > 0.$$

Then,

$$|\Delta_{n-1,l,m}(z_j; f)| \leq \left(\frac{1}{\rho^{\beta m}} - \eta\right)^n$$

for large n.Hence

$$\begin{aligned} h(z_j) &= \Delta_{n-1,l,m}(z_j; f) - z_j^{\beta m} \Delta_{n,l,m}(z_j; f) \\ &= \mathcal{O}\left(\frac{1}{\rho^{\beta m}} - \eta\right)^n \end{aligned}$$

hence from (2.3.24) we obtain

$$\sum_{k=0}^{\beta m-1} a_{k+\beta mn} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{\beta m}} - \eta\right)^n . \tag{2.3.25}$$

Similarly for $j > \mu$ from (2.3.23), (2.3.8) and (2.3.9), we have

$$\begin{aligned} h(z_j) &= -\sum_{k=0}^{\beta m} a_{k+(\beta m+1)n} z_j^{k+n} + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{\beta mn}} + \frac{|z_j|^n}{(\rho-\epsilon)^{((\beta+1)m+1)n}}\right) \\ &= -\sum_{k=0}^{\beta m} a_{k+(\beta m+1)n} z_j^{k+n} + \mathcal{O}\left(\frac{|z_j|}{\rho^{\beta m+1}} - \eta\right)^n . \end{aligned} \tag{2.3.26}$$

Since from hypothesis $B_{l,m}(z_j; f) < K_{\beta,m}(|z_j|, \rho)$ $(j = \mu+1, \ldots, q)$, i.e.,

$$\overline{\lim_{n\to\infty}} |\Delta_{n-1,l,m}(z_j; f)|^{1/n} < \frac{|z_j|}{\rho^{1+\beta m}}.$$

We can put

$$\overline{\lim_{n\to\infty}} |\Delta_{n-1,l,m}(z_j; f)|^{1/n} = \frac{|z_j|}{\rho^{1+\beta m}} - \eta, \eta > 0$$

that is,

$$|\Delta_{n-1,l,m}(z_j; f)| \leq \left(\frac{|z_j|}{\rho^{1+\beta m}} - \eta\right)^n$$

for large n. Thus

$$\begin{aligned} h(z_j) &= \Delta_{n-1,l,m}(z_j;f) - z_j^{\beta m}\Delta_{n,l,m}(z_j;f) \\ &= \mathcal{O}\left(\frac{|z_j|}{\rho^{\beta m+1}} - \eta\right)^n. \end{aligned}$$

Hence from (2.3.26) we obtain

$$\sum_{k=0}^{\beta m} a_{k+(\beta m+1)n} z_j^{k+n} = \mathcal{O}\left(\frac{|z_j|}{\rho^{\beta m+1}} - \eta\right)^n$$

or,

$$\sum_{k=0}^{\beta m} a_{k+(\beta m+1)n} z_j^{k} = \mathcal{O}\left(\frac{1}{\rho^{\beta m+1}} - \eta\right)^n \tag{2.3.27}$$

for large n.

Now, since (2.3.25) and (2.3.27) holds for all n put $n = (\beta m+1)\nu + \lambda, \lambda = 0, \ldots, \beta m$ in (2.3.25) and $n = \beta m\nu + \lambda, \lambda = 0, \ldots, \beta m - 1$ in (2.3.27) we have

$$\sum_{k=0}^{\beta m-1} a_{k+\beta m(\beta m+1)\nu+\lambda\beta m} z_j^{k} = \mathcal{O}\left(\frac{1}{\rho^{\beta m}} - \eta\right)^{(\beta m+1)\nu+\lambda} \tag{2.3.28}$$

$(j = 1, \ldots, \mu; \lambda = 0, 1, \ldots \beta m; \nu = 0, 1, \ldots)$,

$$\sum_{k=0}^{\beta m} a_{k+(\beta m+1)\beta m\nu+(\beta m+1)\lambda} z_j^{k} = \mathcal{O}\left(\frac{1}{\rho^{\beta m+1}} - \eta\right)^{\beta m\nu+\lambda} \tag{2.3.29}$$

$(j = \mu+1, \ldots, q; \lambda = 0, 1, \ldots \beta m - 1; \nu = 0, 1, \ldots)$.

Now since

$$\frac{1}{\rho^{\beta m}} - \eta < \frac{1}{\rho^{\beta m}}, \qquad \eta > 0$$

so,

$$\left(\frac{1}{\rho^{\beta m}} - \eta\right)^{\beta m+1} < \frac{1}{\rho^{\beta m(\beta m+1)}}.$$

Choose η_1 such that

$$0 < \eta_1 < \frac{1}{\rho^{\beta m(\beta m+1)}} - \left(\frac{1}{\rho^{\beta m}} - \eta\right)^{\beta m+1}$$

or,

$$\left(\frac{1}{\rho^{\beta m}} - \eta\right)^{(\beta m+1)\nu} < \left(\frac{1}{\rho^{\beta m(\beta m+1)}} - \eta_1\right)^{\nu}. \tag{2.3.30}$$

Hence (2.3.28) can be written as

$$\sum_{k=0}^{\beta m-1} a_{k+\beta m(\beta m+1)\nu+\lambda\beta m} z_j^{k} = \mathcal{O}\left(\frac{1}{\rho^{\beta m(\beta m+1)}} - \eta\right)^{\nu} \tag{2.3.31}$$

$(j = 1, \ldots, \mu; \lambda = 0, 1, \ldots \beta m; \nu = 0, 1, \ldots)$.

Similarly (2.3.29) can be written as

$$\sum_{k=0}^{\beta m} a_{k+(\beta m+1)\beta m\nu+(\beta m+1)\lambda} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{\beta m(\beta m+1)}} - \eta\right)^\nu \tag{2.3.32}$$

$(j = \mu + 1, \ldots, q; \lambda = 0, 1, \ldots \beta m - 1; \nu = 0, 1, \ldots)$.

Note that (2.3.31) and (2.3.32) can be written as

$$M.A^T = B \tag{2.3.33}$$

where $A = \left(a_{\beta m(\beta m+1)\nu}, a_{\beta m(\beta m+1)\nu+1}, \ldots, a_{\beta m(\beta m+1)\nu+\beta m(\beta m+1)-1}\right)$ and $B = \left(\mathcal{O}(\frac{1}{\rho^{\beta m(\beta m+1)}} - \eta)^\nu\right)$, B is a column vector of order $(q\beta m + \mu) \times 1$.

Since $rank M = \beta m(\beta m + 1)$, solving (2.3.33) we get

$$a_{\beta m(\beta m+1)\nu+k} = \mathcal{O}\left(\frac{1}{\rho^{\beta m(\beta m+1)}} - \eta\right)^\nu$$

for $k = 0, 1, \ldots, \beta m(\beta m + 1) - 1$. Hence

$$\overline{\lim_{\nu\to\infty}} |a_\nu|^{1/\nu} < \frac{1}{\rho}$$

which is a contradiction to $f \in A_\rho$.

Remark 2.3.1 For $m = 1$ Theorem 2.3.1 reduces to Theorem 2.1.5.

Corollary 2.3.1 *If either $\mu \geq \beta m$, or $q - \mu \geq \beta m + 1$ (that is, there are either at least βm points in $|z| < \rho$ or at least $\beta m + 1$ points in $|z| > \rho$), then Z is not a (β, m, ρ)-distinguished set.*

Proof Since if $\mu \geq \beta m$, we take the minor of M which consist of the first βm rows of each X in M. Its determinant is $(V(z_1, \ldots, z_{\beta m}))^{\beta m+1} \neq 0$, where $V(z_1, \ldots, z_{\beta m})$ is the Vandermond determinant. Similar reason applies for $q - \mu \geq \beta m + 1$.

Corollary 2.3.2 *If $\mu < q \leq \beta m$, or $\mu = q < \beta m$, then Z is an (β, m, ρ)-distinguished set.*

This follows from the fact that number of rows in M is $q\beta m + \mu < \beta m(\beta m + 1)$ so that $rank M < \beta m(\beta m + 1)$.

Remark 2.3.2 Corollary 2.3.1 gives the following :

Theorem 2.3.2 *Let $f \in A_\rho, \rho > 1$ and $l \geq 1$ with β the smallest positive integer such that $\beta m > l-1$. Then*

(i) $$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,m}(z;f) \mid^{1/n} = \frac{\mid z \mid}{\rho^{1+\beta m}}$$

for all but at most βm distinct points in $\mid z \mid > \rho$.

(ii) $$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,m}(z;f) \mid^{1/n} = \frac{1}{\rho^{\beta m}}$$

for all but at most $\beta m - 1$ distinct points in $0 < \mid z \mid < \rho$.

Remark 2.3.3 For $m = 1$ Theorem 2.3.2 reduces to Theorem 2.1.3.

From Theorem 2.2.1 and Theorem 2.3.2 we have

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,m}(z;f) \mid^{1/n} < \frac{\mid z \mid}{\rho^{1+\beta m}}$$

for at most βm distinct points in $\mid z \mid > \rho$. That is for $|z| > \rho^{1+\beta m}$

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,m}(z;f) \mid^{1/n} < B, \;\; B > 1$$

for at most βm distinct points. In other words we can say that

Remark 2.3.4 Let $f \in A_\rho, \rho > 1$ and $l \geq 1$ with β the smallest positive integer such that $\beta m > l-1$ then the sequence $\{\Delta_{n-1,l,m}(z;f)\}_{n=1}^{\infty}$ can be bounded at most at βm distinct points in $|z| > \rho^{1+\beta m}$.

Corollary 2.3.3 *If f is analytic on $|z| \leq 1$ and if $\Delta_{n-1,l,m}(z;f)$ is uniformly bounded in every closed subdomain of $|z| < \rho^{1+\beta m}$ then f is analytic in $|z| < \rho$.*

Proof If f is analytic on $|z| \leq 1$. Let $f \in A_{\rho_1}$ where $\rho_1 > 1$, then from Theorem 2.2.1, $g_{\beta,m} = K_{\beta,m}(R, \rho_1)$. Thus, by above Remark 2.3.4 $\{\Delta_{n-1,l,m}(z;f)\}_{n=1}^{\infty}$ can be bounded at most at βm distinct points in $|z| > {\rho_1}^{1+\beta m}$. Also it is given that $\Delta_{n-1,l,m}(z;f)$ is uniformly bounded in every closed subdomain of $|z| < \rho^{1+\beta m}$. Hence $\rho_1 < \rho$ is not possible. That is $\rho_1 \geq \rho$ which gives that f is analytic in $|z| < \rho$.

Remark 2.3.5 When $\mu = 0, q = \beta m$ or $\mu = q = \beta m - 1$ Corollary 2.3.2 implies the following :

Theorem 2.3.3 *Let $\rho > 1$ and $l \geq 1$ with β the smallest positive integer such that $\beta m > l - 1$.*

(i) If $z_1, \ldots, z_{\beta m}$ are arbitrary βm points with modulus greater than ρ then there is a rational function $f \in A_\rho$ with

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,m}(z; f) \mid^{1/n} < \frac{\mid z_j \mid}{\rho^{1+\beta m}}, \qquad j = 1, \ldots, \beta m.$$

(ii) If $z_1, \ldots, z_{\beta m-1}$ are arbitrary $\beta m - 1$ points in the ring $0 < |z| < \rho$ then there is a rational function $f \in A_\rho$ with

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,m}(z; f) \mid^{1/n} = \frac{1}{\rho^{\beta m}}, \qquad j = 1, \ldots, \beta m - 1.$$

Remark 2.3.6 For $m = 1$ Theorem 2.3.3 reduces to Theorem 2.1.4.

2.4 In this section we study $\Delta^{(r)}_{n-1,l,m}(z; f)$ which is the r^{th} derivative of $\Delta_{n-1,l,m}(z; f)$ with respect to z. We give some quantitative estimates for $\overline{\lim}_{n\to\infty} \max_{|z|=R} |\Delta^{(r)}_{n-1,l,m}(z; f)|^{1/n}$ and by introducing the concept of distinguished point of degree r we investigated some relations between the order of pointwise convergence of $\Delta^{(r)}_{n-1,l,m}(z; f)$ and the properties of $f(z)$.

Theorem 2.4.1 *For each $f \in R_\rho(\rho > 1)$, for any integer $l \geq 1$ let β be the least positive integer such that $\beta m > l - 1$ and $r \geq 0$, and any $R > 0$, there holds*

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta^{(r)}_{n-1,l,m}(z; f)|^{1/n} \leq K_{\beta,m}(R, \rho) \tag{2.4.1}$$

where $K_{\beta,m}(R, \rho)$ is given by (2.2.1). Equality holds in (2.4.1) iff $f \in A_\rho$.

Proof : Set $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then for every z on $|z| = R(R > 0)$ we have from (2.2.3)

$$\begin{aligned}
\Delta^{(r)}_{n-1,l,m}(z; f) &= \left(\sum_{k=0}^{n-1}\sum_{j=\beta}^{\infty} a_{k+njm} z^k\right)^{(r)} \\
&= \sum_{k=r}^{n-1}\sum_{j=\beta}^{\infty} (k)_r a_{k+njm} z^{k-r} \\
&= \mathcal{O}\left(\sum_{k=0}^{n-1}\sum_{j=\beta}^{\infty} (k)_r \frac{|z|^{k-r}}{(\rho-\epsilon)^{njm+k}}\right) \\
&= \mathcal{O}\begin{cases} n^r \frac{R^n}{(\rho-\epsilon)^{\beta mn+n}} & \text{if} \quad R \geq \rho \\ n^r \frac{1}{(\rho-\epsilon)^{\beta mn}} & \text{if} \quad 0 < R < \rho \end{cases}
\end{aligned}$$

where $(k)_r = k(k-1)\dots(k-r+1)$, $(k)_0 := 1$. Hence

$$|\Delta^{(r)}_{n-1,l,m}(z;f)| \le M \begin{cases} \left(\frac{n^r R^n}{(\rho-\epsilon)^{\beta mn+n}}\right) & \text{if} \quad R \ge \rho \\ \left(\frac{n^r}{(\rho-\epsilon)^{\beta mn}}\right) & \text{if} \quad 0 < R < \rho \end{cases}$$

thus,

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta^{(r)}_{n-1,l,m}(z;f)|^{1/n} \le \begin{cases} \frac{R}{(\rho-\epsilon)^{1+\beta m}} & \text{if} \quad R \ge \rho \\ \frac{1}{(\rho-\epsilon)^{\beta m}} & \text{if} \quad 0 < R < \rho \end{cases}$$

since ϵ is arbitrary small, we obtain (2.4.1).

To prove the second part we show that equality does not hold in (2.4.1) iff $f \in R_\rho \backslash A_\rho$. First suppose equality does not hold in (2.4.1), then there is some $r' \ge 0$ and $f \in R_\rho$ for which strict inequality holds in (2.4.1).That is

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta^{(r')}_{n-1,l,m}(z;f)|^{1/n} < K_{\beta,m}(R,\rho). \tag{2.4.2}$$

Thus

$$\Delta^{(r')}_{n-1,l,m}(z;f) = \sum_{k=r'}^{n-1} (k)_{r'} a_{k+\beta mn} z^{k-r'} + \sum_{k=r'}^{n-1} \sum_{j=\beta+1}^{\infty} (k)_{r'} a_{k+njm} z^{k-r'}$$

Let $R \ge \rho$, then

$$\begin{aligned} \sum_{k=n-lm-1}^{n-1} (k)_{r'} a_{k+\beta mn} z^{k-r'} &= \Delta^{(r')}_{n-1,l,m}(z;f) - \sum_{k=r'}^{n-\beta m-2} (k)_{r'} a_{k+\beta mn} z^{k-r'} \\ &\quad - \sum_{k=r'}^{n-1} \sum_{j=\beta+1}^{\infty} (k)_{r'} a_{k+njm} z^{k}. \end{aligned}$$

By Cauchy integral formula, we have for $n - \beta m - 1 \le k \le n-1$

$$\begin{aligned} (k)_{r'} a_{k+\beta mn} &= \frac{1}{2\pi i} \int_{|z|=R} \frac{\Delta^{(r')}_{n-1,l,m}(z;f)}{z^{k-r'+1}} dz - \\ &\quad - \sum_{k'=r'}^{n-\beta m-2} (k')_{r'} a_{k'+\beta mn} \frac{1}{2\pi i} \int_{|z|=R} \frac{z^{k'-r'}}{z^{k-r'+1}} dz \\ &\quad - \frac{1}{2\pi i} \int_{|z|=R} \frac{\sum_{k'=r'}^{n-1} \sum_{j=\beta+1}^{\infty} a_{k'+njm}}{z^{k-r'+1}} dz \\ &\le \max_{|z|=R} |\Delta^{(r')}_{n-1,l,m}(z;f)| R^{-k} + 0 + \mathcal{O}\left(n^{r'} R^{-k} \frac{R^n}{(\rho-\epsilon)^{n+(\beta+1)mn}}\right). \end{aligned}$$

Hence from (2.4.2)

$$\overline{\lim_{n\to\infty}} |a_{k+\beta mn}|^{k+\beta mn} < max \left\{ \left(R^{-1} \frac{R}{\rho^{\beta m+1}}\right)^{\frac{1}{\beta m+1}}, \left(R^{-1} \frac{R}{(\rho-\epsilon)^{(\beta+1)m+1}}\right)^{\frac{1}{\beta m+1}} \right\}.$$

By choosing $\epsilon > 0$ sufficiently small so that

$$(\rho-\epsilon)^{-((\beta+1)m+1)} < \rho^{-(\beta m+1)}$$

we have

$$\overline{\lim_{n\to\infty}}|a_{k+\beta mn}|^{\frac{1}{k+\beta mn}} < \frac{1}{\rho}, \qquad n-\beta m-1 \le k \le n-1$$

or,

$$\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}} < \frac{1}{\rho}.$$

Thus we have $f \in R_\rho \backslash A_\rho$.

Similarly for $R < \rho$ by taking $0 \le k \le \beta m - 1$ from (2.4.2) we have

$$\begin{aligned}\overline{\lim_{n\to\infty}}|a_{k+\beta mn}|^{\frac{1}{k+\beta mn}} &< max\left\{\left(\frac{1}{\rho^{\beta m}}\right)^{\frac{1}{\beta m}}, \left(\frac{1}{(\rho-\epsilon)^{(\beta+1)m}}\right)^{\frac{1}{\beta m}}\right\}\\ &= \frac{1}{\rho}, \qquad 0 \le k \le \beta m - 1\end{aligned}$$

or,

$$\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}} < \frac{1}{\rho}.$$

Thus we have $f \in R_\rho \backslash A_\rho$.

Next, let $f \in R_\rho \backslash A_\rho$. Thus $f \in R_{\rho_1}$ for some $\rho_1 > \rho$, hence by the first part of Theorem 2.4.1 we have

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta^{(r)}_{n-1,l,m}(z;f)|^{1/n} \le K_{\beta,m}(R,\rho_1).$$

Since $\rho_1 > \rho$ hence by definition

$$K_{\beta,m}(R,\rho_1) < K_{\beta,m}(R,\rho),$$

with above equation which gives

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta^{(r)}_{n-1,l,m}(z;f)|^{1/n} < K_{\beta,m}(R,\rho).$$

Thus equality does not hold in (2.4.1) if $f \in R_\rho \backslash A_\rho$.

Corollary 2.4.1 *For each $f \in R_\rho(\rho > 1)$, and any integer $l \ge 1, r \ge 0$, there holds*

$$\lim_{n\to\infty} \Delta^{(r)}_{n-1,l,m}(z;f) = 0, \qquad \forall\ |z| < \rho^{1+\beta m}.$$

Moreover the result is best possible if $f \in A_\rho$.

Remark 2.4.1 For $r = 0$ Theorem 2.4.1 gives Theorem 2.2.1.

Remark 2.4.2 For $m = 1$ Theorem 2.4.1 gives Theorem 2.1.6.

For any integer $r \geq 0$, we set

$$H^r_{\beta,m}(z; f) := \overline{\lim_{n\to\infty}}|\Delta^{(r)}_{n-1,l,m}(z; f)|^{1/n}.$$

We say that η is an (β, m, ρ)-distinguished point of $f \in A_\rho$ of degree r if

$$H^\nu_{\beta,m}(\eta; f) < K_{\beta,m}(|\eta|, \rho), \qquad \forall\ \nu = 0, 1, \ldots, r-1,$$

and consider it as r points coincided at η.

Hereafter let $\{\eta_\nu\}_{\nu=1}^s$ be a set of s points in C and p_ν denote the number of appearence of η_ν in $\{\eta_j\}_{j=1}^\nu$. We prove

Theorem 2.4.2 *If $f \in R_\rho (\rho > 1), l$ is any positive integer, for which β is the least positive integer such that $\beta m > l - 1$ and there are $\beta m + 1$ points $\{\eta_\nu\}_{\nu=1}^{\beta m+1}$ in $|z| > \rho$ (or, βm points $\{\eta_\nu\}_{\nu=1}^{\beta m}$ in $|z| < \rho$) for which*

$$H^{p_\nu-1}_{\beta,m}(\eta_\nu; f) < K_{\beta,m}(|\eta_\nu|, \rho), \qquad \nu = 1, \ldots, \beta m + 1 (or\ \beta m),$$

then $f \in R_\rho \backslash A_\rho$.

For the proof of Theorem 2.4.2, we need

Lemma 2.4.1 *Let $g(z) = \sum_{k=0}^\infty a_k z^k \in R_\rho(\rho > 1), l$ be any positive integer and $w_s(z) := \prod_{\nu=1}^s (z - \eta_\nu) = \sum_{k=0}^s C_k z^k$, where $\{\eta_\nu\}_{\nu=1}^s$ are any given s points in $|z| > \rho$ (or in $|z| < \rho$), then*

$$H^{p_\nu-1}_{\beta,m}(\eta_\nu; w_s g) < K_{\beta,m}(|\eta_\nu|, \rho), \qquad \nu = 1, \ldots, s \tag{2.4.3}$$

iff there is a $\rho_0 > \rho$ such that for $\nu = 1, 2, \ldots, s$

$$a_{(\beta m+1)n-\nu} = \mathcal{O}\left(\rho_0^{-(\beta m+1)n}\right) \left(or\ a_{\beta mn-\nu} = \mathcal{O}(\rho_0^{-\beta mn})\right).$$

proof : From (2.2.3)

$$\Delta_{n-1,l,m}(z, g) = \sum_{k=0}^{n-1}\sum_{j=\beta}^{\infty} a_{k+njm} z^k.$$

Similarly, for any positive integer ν, we have

$$\Delta_{n-1,l,m}(z, z^\nu g) = \sum_{k=0}^{n-1}\sum_{j=\beta}^{\infty} a_{k-\nu+njm} z^k.$$

According to the linearity property of $\Delta_{n-1,l,m}(z; f)$ it follows that

$$\begin{aligned}
\Delta_{n-1,l,m}(z, \omega_s g) &= \sum_{\nu=0}^{s} C_\nu \sum_{k=0}^{n-1}\sum_{j=\beta}^{\infty} a_{k-\nu+njm} z^k \\
&= \sum_{\nu=0}^{s} C_\nu \sum_{k=-\nu}^{n-\nu-1}\sum_{j=\beta}^{\infty} a_{k+njm} z^{k+\nu} \\
&= \sum_{j=\beta}^{\infty}\sum_{\nu=0}^{s} C_\nu \left(\sum_{k=-\nu}^{-1} + \sum_{k=0}^{n-1} - \sum_{k=n-\nu}^{n-1}\right) a_{k+njm} z^{k+\nu} \\
&= \sum_{j=\beta}^{\infty}\sum_{\nu=1}^{s} C_\nu \sum_{k=0}^{\nu-1} a_{k-\nu+njm} z^k + \omega_s(z)\Delta_{n-1,l,m}(z; f) \\
&\quad - z^n \sum_{j=\beta}^{\infty}\sum_{\nu=1}^{s} C_\nu \sum_{k=0}^{\nu-1} a_{k+n-\nu+njm} z^k. \qquad (2.4.4)
\end{aligned}$$

Next, we have

$$\begin{aligned}
\sum_{j=\beta}^{\infty}\sum_{\nu=1}^{s} C_\nu \sum_{k=0}^{\nu-1} a_{k-\nu+njm} z^k &= \sum_{j=\beta}^{\infty}\sum_{k=0}^{s-1} z^k \sum_{\nu=k+1}^{s} C_\nu a_{k-\nu+njm} \\
&= \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{njm-\nu}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\sum_{j=\beta}^{\infty}\sum_{\nu=1}^{s} C_\nu \sum_{k=0}^{\nu-1} a_{k+n-\nu+njm} z^k &= \sum_{j=\beta}^{\infty}\sum_{k=0}^{s-1} z^k \sum_{\nu=k+1}^{s} C_\nu a_{k+n-\nu+njm} \\
&= \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{n+njm-\nu}.
\end{aligned}$$

Substituting these in (2.4.4) we have

$$\begin{aligned}
\Delta_{n-1,l,m}(z, \omega_s g) &= \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{njm-\nu} + \omega_s(z)\Delta_{n-1,l,m}(z; g) \\
&\quad - z^n \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{n+njm-\nu} \\
&= \omega_s(z)\Delta_{n-1,l,m}(z; g) + \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{jmn-\nu} \\
&\quad - z^n \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{jmn+n-\nu}. \qquad (2.4.5)
\end{aligned}$$

Now, since η_ν occurs p_ν times in $\{\eta_j\}_{j=1}^{\nu}$ hence $\omega_s^{(r)}(z) = 0$ at $z = \eta_\nu$ and $r = 0, \ldots p_\nu - 1$. Thus, for $\nu = 1, 2, \ldots, s$ we have

$$\begin{aligned}\Delta_{n-1,l,m}^{(p_\nu-1)}(\eta_\nu, w_s g) &= 0 + \sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} z^{k-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{jmn-\nu} \\ &\quad - \sum_{k=0}^{s-1} (k+n)_{p_\nu-1} z^{k+n-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{jmn+n-\nu}. \end{aligned} \tag{2.4.6}$$

If points are in $|z| > \rho$ then

$$\Delta_{n-1,l,m}^{(p_\nu-1)}(\eta_\nu, w_s g) = \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{\beta mn}} + \frac{|\eta_\nu|^n}{\rho_0^{(\beta m+1)n}}\right).$$

Now, for $|\eta_\nu| > \rho$, for a given $\epsilon > 0$ we can find $\eta > 0$ such that

$$\frac{|\eta_\nu|^n}{\rho_0^{(\beta m+1)n}} \leq \left(\frac{|\eta_\nu|}{\rho^{\beta m+1}} - \eta\right)^n, \qquad \rho_0 > \rho$$

and from (2.3.8)

$$\frac{1}{(\rho-\epsilon)^{\beta mn}} \leq \left(\frac{|\eta_\nu|}{\rho^{\beta m+1}} - \eta\right)^n, \qquad |\eta_\nu| > \rho.$$

Thus,

$$\Delta_{n-1,l,m}^{(p_\nu-1)}(\eta_\nu, w_s g) = \mathcal{O}\left(\frac{|\eta_\nu|}{\rho^{\beta m+1}} - \eta\right)^n,$$

hence

$$H_{\beta,m}^{p_\nu-1}(\eta_\nu; w_s g) < \frac{|\eta_\nu|}{\rho^{\beta m+1}}, \qquad \nu = 1, 2, \ldots, s.$$

Similarly if points are in $|z| < \rho$ then

$$\Delta_{n-1,l,m}^{(p_\nu-1)}(\eta_\nu, w_s g) = \mathcal{O}\left(\frac{1}{\rho_0^{\beta mn}} + \frac{|\eta_\nu|^n}{(\rho-\epsilon)^{(\beta m+1)n}}\right).$$

For $|\eta_\nu| < \rho$, for a given $\epsilon > 0$ we can find $\eta > 0$ such that

$$\frac{1}{\rho_0^{\beta mn}} \leq \left(\frac{1}{\rho^{\beta m}} - \eta\right)^n, \qquad \rho_0 > \rho$$

and from (2.3.21)

$$\frac{|\eta_\nu|^n}{(\rho-\epsilon)^{(1+\beta m)n}} \leq \left(\frac{1}{\rho^{\beta m}} - \eta\right)^n, \qquad |\eta_\nu| < \rho.$$

Thus,

$$\Delta_{n-1,l,m}^{(p_\nu-1)}(\eta_\nu, w_s g) = \mathcal{O}\left(\frac{1}{\rho^{\beta m}} - \eta\right)^n,$$

hence

$$H_{\beta,m}^{p_\nu-1}(\eta_\nu; w_s g) < \frac{1}{\rho^{\beta m}}, \qquad \nu = 1, 2, \ldots, s.$$

Conversely, suppose (2.4.3) is valid. First let us consider when $\{\eta_\nu\}_{\nu=1}^s$ are in $|z| > \rho$. Since $g \in R_\rho$, by continuity there is a $\rho_1 > \rho$ with

$$\rho < \rho_1 < min\left[\rho^{((\beta+1)m+1)/(\beta m+1)}, (\rho^{\beta m} min_{1\le\nu\le s}|\eta_\nu|)^{1/(\beta m+1)}\right]$$

such that

$$H_{\beta,m}^{p_\nu-1}(\eta_\nu; w_s g) < K_{\beta,m}(|\eta_\nu|, \rho_1), \qquad \nu = 1, \ldots, s. \tag{2.4.7}$$

From (2.4.5)

$$\begin{aligned}
&\sum_{k=r}^{s-1}(k)_r z^{k-r}\sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn+n-\nu}\\
&= \left(\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn+n-\nu}\right)^{(r)}\\
&= \sum_{b=0}^{r}(\binom{r}{b})\left(\frac{\omega_s(z)}{z^n}\right)^{(b)}\Delta_{n-1,l,m}^{(r-b)}(z;g)-\\
&-\sum_{b=0}^{r}(\binom{r}{b})(z^{-n})^{(b)}\Delta_{n-1,l,m}^{(r-b)}(z;\omega_s g)+\\
&+\sum_{b=0}^{r}(\binom{r}{b})(z^{-n})^{(b)}\left(\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn-\nu}\right)^{(r-b)}.
\end{aligned}$$

On taking $r = p_\nu - 1$ and $z = \eta_\nu (\nu = 1, \ldots, s)$, from (2.4.7) and the fact that $f \in R_\rho$ and the definition of p_ν we have

$$\begin{aligned}
&\textstyle\sum_{k=p_\nu-1}^{s-1}(k)_{p_\nu-1}{\eta_\nu}^{k-p_\nu+1}\sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn+n-\nu}\\
&= 0 + \mathcal{O}\left(n^{p_\nu-1}|\eta_\nu|^{-n}(K_{\beta,m}(|\eta_\nu|,\rho_1))^n\right. + \mathcal{O}\left(n^{p_\nu-1}|\eta_\nu|^{-n}(\rho-\epsilon)^{-\beta mn}\right)\\
&= \mathcal{O}\left(n^{p_\nu-1} max\left(\frac{1}{\rho_1^{(\beta m+1)n}}, \frac{1}{min|\eta_\nu|^n(\rho-\epsilon)^{\beta mn}}\right)\right).
\end{aligned}$$

Since ϵ is arbitrary small hence by the choice of ρ_1 we have

$$\begin{aligned}
&\sum_{k=p_\nu-1}^{s-1}(k)_{p_\nu-1}{\eta_\nu}^{k-p_\nu+1}\sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn+n-\nu}\\
&= \mathcal{O}\left(n^{p_\nu-1}{\rho_1}^{-(\beta m+1)n}\right), \qquad \nu = 1, 2, \ldots, s.
\end{aligned} \tag{2.4.8}$$

Since η_ν are all distinct thus on solving (2.4.8) we have

$$\sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn+n-\nu} = \mathcal{O}\left(n^\tau {\rho_1}^{-(\beta m+1)n}\right), \tag{2.4.9}$$

where $\tau = \max_{1\le\nu\le s}[p_\nu - 1], k = 0, 1, \ldots, s-1$. Solving (2.4.9) we have

$$\sum_{j=\beta}^{\infty} a_{jmn+n-\nu} = \mathcal{O}\left(n^\tau {\rho_1}^{-(\beta m+1)n}\right), \qquad \nu = 1, 2, \ldots, s,$$

so that by the choice of ρ_1

$$\begin{aligned}
a_{\beta mn+n-\nu} &= \mathcal{O}\left(n^{\tau}\rho_1^{-(\beta m+1)n}\right) - \sum_{j=\beta+1}^{\infty} a_{jmn+n-\nu} \\
&= \mathcal{O}\left(n^{\tau}\rho_1^{-(\beta m+1)n}\right) + \mathcal{O}\left((\rho-\epsilon)^{-((\beta+1)m+1)n}\right) \\
&= \mathcal{O}\left(n^{\tau}\rho_1^{-(\beta m+1)n}\right) \\
&= \mathcal{O}\left(\rho_0^{-(\beta m+1)n}\right)
\end{aligned}$$

where $\rho_0 \in (\rho, \rho_1)$.

In the case when $\{\eta_\nu\}_{\nu=1}^{s}$ are in $|z| < \rho$, since $g \in R_\rho$, by continuity there is a $\rho_1 > \rho$ with

$$\rho < \rho_1 < min\left[\rho^{((\beta+1)m)/(\beta m)}, (\rho^{\beta m+1} min_{1\le\nu\le s}|\eta_\nu|^{-1})^{1/\beta m}\right]$$

such that

$$H_{\beta,m}^{p_\nu-1}(\eta_\nu; w_s g) < K_{\beta,m}(|\eta_\nu|, \rho_1), \qquad \nu = 1, \dots, s. \tag{2.4.10}$$

From (2.4.5)

$$\begin{aligned}
\sum_{k=r}^{s-1}(k)_r z^{k-r}\sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn-\nu} &= \left(\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn-\nu}\right)^{(r)} \\
&= \sum_{b=0}^{r}(\binom{r}{b})\,(\omega_s(z))^{(b)}\,\Delta_{n-1,l,m}^{(r-b)}(z;g) - \Delta_{n-1,l,m}^{(r)}(z;\omega_s g) \\
&\quad + \left(\sum_{k=0}^{s-1} z^{k+n}\sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn+n-\nu}\right)^{(r)}.
\end{aligned}$$

On taking $r = p_\nu - 1$ and $z = \eta_\nu (\nu = 1, \dots, s)$, from (2.4.10) and the fact that $f \in R_\rho$ and the definition of p_ν we have

$$\begin{aligned}
\sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1}\eta_\nu^{k-p_\nu+1}\sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn-\nu} &= 0 + \mathcal{O}\left(n^{p_\nu-1}(K_{\beta,m}(|\eta_\nu|, \rho_1))\right)^n \\
+\mathcal{O}\left(n^{p_\nu-1}\frac{|\eta_\nu|^n}{(\rho-\epsilon)^{-\beta mn-n}}\right) \\
&= \mathcal{O}\left(n^{p_\nu-1} max\left(\frac{1}{\rho_1^{\beta mn}}, \frac{max|\eta_\nu|^n}{(\rho-\epsilon)^{\beta mn+n}}\right)\right).
\end{aligned}$$

Since ϵ is arbitrary small hence by the choice of ρ_1 we have

$$\sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1}\eta_\nu^{k-p_\nu+1}\sum_{\nu=1}^{s-k} C_{\nu+k}\sum_{j=\beta}^{\infty} a_{jmn-\nu}$$

$$= \mathcal{O}\left(n^{p_\nu-1}\rho_1^{-\beta mn}\right), \qquad \nu = 1, 2, \dots, s. \tag{2.4.11}$$

Since η_ν are all distinct thus on solving (2.4.11) we have

$$\sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} a_{jmn-\nu} = \mathcal{O}\left(n^\tau \rho_1^{-\beta mn}\right), \tag{2.4.12}$$

where $\tau = \max_{1\le\nu\le s}[p_\nu - 1], k = 0, 1, \ldots, s-1$. Solving (2.4.12) we have

$$\sum_{j=\beta}^{\infty} a_{jmn-\nu} = \mathcal{O}\left(n^\tau \rho_1^{-\beta mn}\right), \qquad \nu = 1, 2, \ldots, s,$$

so that by the choice of ρ_1

$$\begin{aligned}
a_{\beta mn-\nu} &= \mathcal{O}\left(n^\tau \rho_1^{-\beta mn}\right) - \sum_{j=\beta+1}^{\infty} a_{jmn-\nu} \\
&= \mathcal{O}\left(n^\tau \rho_1^{-\beta mn}\right) + \mathcal{O}\left((\rho-\epsilon)^{-((\beta+1)mn)}\right) \\
&= \mathcal{O}\left(n^\tau \rho_1^{-\beta mn}\right) \\
&= \mathcal{O}\left(\rho_0^{-\beta mn}\right)
\end{aligned}$$

where $\rho_0 \in (\rho, \rho_1)$, which completes the proof.

Proof of Theorem 2.4.2 : For points in $|z| > \rho$ let

$$g(z) = \frac{f(z)}{\prod_{\nu=1}^{\beta m+1}(z-\eta_\nu)} = \sum_{k=0}^{\infty} a_k z^k,$$

thus $f(z) = w_{\beta m+1}(z) g(z)$ and $g \in R_\rho$. According to Lemma 2.4.1, there is a $\rho_0 > \rho$ such that

$$a_{(\beta m+1)n-\nu} = \mathcal{O}\left(\rho_0^{-(\beta m+1)n}\right) \qquad \nu = 1, 2, \ldots, \beta m + 1,$$

so that

$$\overline{\lim_{k\to\infty}} |a_k|^{1/k} \le \frac{1}{\rho_0} < \frac{1}{\rho},$$

hence, $g \in R_\rho \backslash A_\rho$ which gives $f \in R_\rho \backslash A_\rho$.

If $\{\eta_\nu\}_{\nu=1}^{s}$ are in $|z| < \rho$, then we set

$$g(z) = [f(z) - L_{\beta m-1}(z)] / \prod_{\nu=1}^{\beta m}(z-\eta_\nu) = \sum_{k=0}^{\infty} a_k z^k,$$

where $L_{\beta m-1}(z)$ is the Lagrange interpolating polynomial of $f(z)$ of degree $\beta m - 1$ at $\{\eta_\nu\}_{\nu=1}^{\beta m}$, then we have $f(z) = \omega_{\beta m}(z) g(z) + L_{\beta m-1}(z)$ where $g(z) \in R_\rho$ and $L_{\beta m-1}(z) \in R_\rho \backslash A_\rho$. By analogous arguments as in $|z| > \rho$ we can show that $g \in R_\rho \backslash A_\rho$ so that $f \in R_\rho \backslash A_\rho$.

Remark 2.4.3 For $m = 1$ Theorem 2.4.2 gives Theorem 2.1.7.

Remark 2.4.4 For $p_\nu = 1, \forall \nu$ Theorem 2.4.2 gives Corollary 2.3.1.

Theorem 2.4.3 *Suppose $f \in A_\rho(\rho > 1), l$ is a positive integer, for which β is the least positive integer such that $\beta m > l - 1$ then*

(a) there are at most βm points $\{\eta_\nu\}_{\nu=1}^{\beta m}$ in $|z| > \rho$ with

$$H_{\beta,m}^{p_\nu - 1}(\eta_\nu; f) < K_{\beta,m}(|\eta_\nu|, \rho), \qquad \nu = 1, \ldots, \beta m$$

(b) there are at most $\beta m - 1$ points $\{\eta_\nu\}_{\nu=1}^{\beta m - 1}$ in $|z| < \rho$ with

$$H_{\beta,m}^{p_\nu - 1}(\eta_\nu; f) < K_{\beta,m}(|\eta_\nu|, \rho), \qquad \nu = 1, \ldots, \beta m - 1.$$

(c) The degree of (β, m, ρ) - distinguished point of $f(z)$ is neither greater than βm in $|z| > \rho$ nor greater than $\beta m - 1$ in $|z| < \rho$.

(d) If either z is in $|z| > \rho$ and $r \geq \beta m + 1$ or z is in $|z| < \rho$ and $r \geq \beta m$, then

$$\overline{\lim_{n \to \infty}} \left[\sum_{\nu=0}^{r} |\Delta_{n-1,l,m}^{(\nu)}(z; f)| \right]^{1/n} = K_{\beta,m}(|z|, \rho).$$

Moreover, for given any η in $|z| > \rho$ and $0 \leq r < \beta m + 1$ or for η in $|z| < \rho$ and $0 \leq r < \beta m$, there is an $f \in A_\rho$ for which

$$\overline{\lim_{n \to \infty}} \left[\sum_{\nu=0}^{r} |\Delta_{n-1,l,m}^{(\nu)}(\eta; f)| \right]^{1/n} < K_{\beta,m}(|\eta|, \rho).$$

Clearly Theorem 2.4.3 follows from Theorem 2.4.2 excluding second part of (d) which follows from the following Theorem 2.4.4.

Remark 2.4.5 For $m = 1$ Theorem 2.4.3 gives Theorem 3 [28].

Remark 2.4.6 For $p_\nu = 1 \forall \nu$ (a) and (b) of Theorem 2.4.3 gives Theorem 2.3.2.

Theorem 2.4.4 *Let $f \in A_\rho(\rho > 1), l$ be any positive integer for which β is the least positive integer such that $\beta m > l - 1$ and $\{\eta_\nu\}_{\nu=1}^{s}$ be any s points in $|z| > \rho, s \leq \beta m$ (or in $|z| < \rho, s \leq \beta m - 1$) , with the numbers p_ν of the appearence of η_ν in $\{\eta_j\}_{j=1}^{\nu}$. Then the neccessary and sufficient condition for*

$$H_{\beta,m}^{p_\nu - 1}(\eta_\nu; f) < K_{\beta,m}(|\eta_\nu|, \rho), \qquad \nu = 1, \ldots, s \qquad (2.4.13)$$

is

$$f(z) = w_s(z)G_s(z) + G_0(z) \tag{2.4.14}$$

where $w_s(z) := \prod_{j=1}^{s}(z-\eta_j)$, $G_0(z) \in R_\rho \backslash A_\rho$ *and* $G_s(z) = \sum_{j=0}^{\infty} \alpha_j z^j \in A_\rho$ *with*

$$\alpha_{(\beta m+1)n-\nu} = 0 \ \ (or, \alpha_{\beta mn-\nu} = 0), \qquad \nu = 1, 2, \ldots, s.$$

proof : *Sufficiency.* Suppose $f(z)$ can be expressed as (2.4.14). Since $G_0(z) \in R_\rho \backslash A_\rho$, according to Theorem 2.4.1

$$\Delta^{(p_\nu-1)}_{n-1,l,m}(\eta_\nu; G_0) < \mathcal{O}\left([K_{\beta,m}(|\eta_\nu|, \rho)]^n\right).$$

that is there exists a $\rho_1 > \rho$ such that

$$\Delta^{(p_\nu-1)}_{n-1,l,m}(\eta_\nu; G_0) = \mathcal{O}\left([K_{\beta,m}(|\eta_\nu|, \rho_1)]^n\right). \tag{2.4.15}$$

Using the hypothesis of G_s, from (2.4.6) we have

$$\begin{aligned}
&\Delta^{(p_\nu-1)}_{n-1,l,m}(\eta_\nu; w_s G_s) \\
&= \sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1}\eta_\nu^{k-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} \alpha_{jmn-\nu} + \\
&-\sum_{k=0}^{s-1}(n+k)_{p_\nu-1}\eta_\nu^{k+n-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta}^{\infty} \alpha_{jmn+n-\nu} \\
&= \mathcal{O}((\rho-\epsilon)^{-\beta mn}) + \sum_{k=0}^{s-1}(n+k)_{p_\nu-1}\eta_\nu^{k+n-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=\beta+1}^{\infty} \alpha_{jmn+n-\nu} \\
&= \mathcal{O}((\rho-\epsilon)^{-\beta mn} + \mathcal{O}\left(|\eta_\nu|^n n^{p_\nu-1}(\rho-\epsilon)^{-((\beta+1)m+1)n}\right).
\end{aligned} \tag{2.4.16}$$

By arbitraryness of $\epsilon > 0$, from (2.4.14), (2.4.15) and (2.4.16) we obtain

$$\begin{aligned}
H^{p_\nu-1}_{\beta,m}(\eta_\nu; f) &\leq max\left(K_{\beta,m}(|\eta_\nu|, \rho_1), \rho^{-\beta m}, |\eta_\nu|\rho^{-((\beta+1)m+1)}\right) \\
&< K_{\beta,m}(|\eta_\nu|, \rho).
\end{aligned}$$

Neccessity. Suppose f satisfies (2.4.13). Let for $|z| > \rho$

$$g(z) = f(z)/w_s(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) \in A_\rho.$$

According to Lemma 2.4.1, from (2.4.13) there exists a $\rho_0 > \rho$ such that

$$a_{(\beta m+1)n-\nu} = \mathcal{O}\left(\rho_0^{-(\beta m+1)n}\right), \qquad \nu = 1, \ldots, s.$$

We set

$$\alpha_{(\beta m+1)n-\nu} = \begin{cases} 0, \text{if } \nu = 1, \ldots, s; n = 1, 2, \ldots, \\ a_{(\beta m+1)n-\nu}, \text{if } \nu = s+1, \ldots, \beta m+1; n = 1, 2, \ldots, \end{cases}$$

and $G_s(z) = \sum_{j=0}^{\infty} \alpha_j z^j$, $g_0(z) = g(z) - G_s(z)$. Clearly $G_s \in A_\rho$ with $\alpha_{(\beta m+1)n-\nu} = 0$, $(\nu = 1, \ldots, s)$ and $g_0(z) = \sum_{j=0}^{\infty} \gamma_j z^j$ with

$$\gamma_{(\beta m+1)n-\nu} = \begin{cases} a_{(\beta m+1)n-\nu}, \text{if } \nu = 1, \ldots, s; n = 1, 2, \ldots, \\ 0, \text{if } \nu = s+1, \ldots, \beta m+1; n = 1, 2, \ldots, \end{cases}$$

hence $g_0(z) \in R_{\rho_0}$. Then we have

$$f(z) = w_s(z)g(z) = w_s(z)[G_s(z) + g_0(z)] = w_s(z)G_s(z) + G_0(z),$$

where $G_0(z) = w_s(z)g_0(z) \in R_{\rho_0}$ and since $\rho_0 > \rho$ thus $G_0(z) \in R_\rho \backslash A_\rho$.

In case $\{\eta_\nu\}_{\nu=1}^{s}$ are in $|z| < \rho$, the proof of sufficeincy is similar. For the necessity part we set

$$g(z) = [f(z) - L_{s-1}(z)]/w_s(z),$$

where $L_{s-1}(z)$ is the Lagrange interpolating polynomial of $f(z)$ of degree $s-1$ at $\{\eta_\nu\}_{\nu=1}^{s}$, then we have $f(z) = w_s(z)g(z) + L_{s-1}(z)$ where $g(z) \in A_\rho$ and $L_{s-1}(z) \in R_\rho \backslash A_\rho$. Similarly we can show that $g(z) = G_s(z) + g_0(z)$, where $g_0 \in R_\rho \backslash A_\rho$ and $G_s \in A_\rho$ with $\alpha_{\beta mn-\nu} = 0, (\nu = 1, 2, \ldots, s)$ and obtain (2.4.14).

Corollary 2.4.2 *Let $f \in A_\rho, (\rho > 1), l$ be any positive integer for which β is the least positive integer such that $\beta m > l-1$ and $\{\eta_\nu\}_{\nu=1}^{s}$ be any s distinct points in $|z| > \rho, s \leq \beta m$ (or in $|z| < \rho, s \leq \beta m - 1$). Then the necessary and sufficient condition for*

$$H_{\beta,m}(\eta_\nu; f) < K_{\beta,m}(|\eta_\nu|, \rho) \quad \nu = 1, \ldots, s$$

is

$$f(z) = w_s(z)G_s(z) + G_0(z),$$

where $w_s(z), G_0(z)$ and $G_s(z)$ have the same meanings as in Theorem 2.4.4.

Corollary 2.4.3 *Let $f \in A_\rho, (\rho > 1), l$ be any positive integer for which β is the least positive integer such that $\beta m > l-1$ and η be any given point in $|z| > \rho, s \leq \beta m$ (or in $|z| < \rho, s \leq \beta m - 1$). Then the necessary and sufficient condition for*

$$H^{\nu}_{\beta,m}(\eta; f) < K_{\beta,m}(|\eta|, \rho), \qquad \nu = 1, \ldots, s-1$$

is

$$f(z) = (z-\eta)^s G_s(z) + G_0(z),$$

where $G_0(z)$ and $G_s(z)$ have the same meanings as in Theorem 2.4.4.

Remark 2.4.7 For $m = 1$ Theorem 2.4.4 gives Theorem 2.1.8.

Remark 2.4.8 For $p_\nu = 1 \forall \nu$ Theorem 2.4.4 gives Corolalry 2.4.2 above which is a generalization of Corollary 2.3.2 and hence of Theorem 2.3.3.

2.5 In this section we consider a set containing the points in $|z| < \rho$ and $|z| > \rho$ simultaneosly and generalise result of section 2.3 for the case that the points of $\{z_j\}_1^s$ can be coincided with each other. We call a set $Z = \{\eta_j\}_1^s$ with $|\eta_j| < \rho, j = 1, \ldots, \mu$ and $|\eta_j| > \rho, j = \mu+1, \ldots, s$ and p_ν denoting the number of appearence of η_ν in $\{\eta_j\}_{j=1}^{\nu}, \nu = 1, \ldots, s$. an (β, m, ρ)-distinguished set if there exists an $f \in A_\rho$ such that $H_{\beta,m}^{p_\nu - 1}(\eta_\nu; f) < K_{\beta,m}(|\eta_\nu|, \rho)$, $\nu = 1, \ldots, s$. To determine a criteria whether Z is an (β, m, ρ)-distinguished set or not we define the matrices X, Y and M(X,Y) as follows:

$$X = \begin{pmatrix} 1 & (z)^{(p_1-1)}|_{z=\eta_1} & \cdots & (z^{\beta m-1})^{(p_1-1)}|_{z=\eta_1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (z)^{(p_\mu-1)}|_{z=\eta_\mu} & \cdots & (z^{\beta m-1})^{(p_\mu-1)}|_{z=\eta_\mu} \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & (z)^{(p_{\mu+1}-1)}|_{z=\eta_{\mu+1}} & \cdots & (z^{\beta m})^{(p_{\mu+1}-1)}|_{z=\eta_{\mu+1}} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (z)^{(p_s-1)}|_{z=\eta_s} & \cdots & (z^{\beta m})^{(p_s-1)}|_{z=\eta_s} \end{pmatrix}.$$

The matrices X and Y are of order $(\mu \times \beta m)$ and $(s-\mu) \times (\beta m + 1)$ respectively. Define

$$M = M(X,Y) = \begin{pmatrix} X & & \\ & X & 0 \\ & & \ddots \\ & 0 & X \\ Y & & \\ & Y & 0 \\ & & \ddots \\ & 0 & Y \end{pmatrix},$$

where X occurs $\beta m + 1$ times and Y occurs βm times beginning under the last X. The matrix M is of order $(s\beta m + \mu) \times \beta m(\beta m + 1)$. We now formulate

Theorem 2.5.1 *Suppose $Z = \{z_j\}_1^s$ is a set of points in C such that $|z_j| < \rho$ ($j = 1, \ldots, \mu$) and $|z_j| > \rho$ ($j = \mu+1, \ldots, s$). Then the set Z is (β, m, ρ) distinguished iff*

$$rank \;\; M < \beta m(\beta m + 1). \tag{2.5.1}$$

Proof : First suppose $rank \;\; M < \beta m(\beta m+1)$. Then there exists a non-zero vector $b = (b_0, b_1, \ldots, b_{\beta m(\beta m+1)-1})$ such that

$$M.b^T = 0. \tag{2.5.2}$$

Set

$$\begin{aligned} f(z) &= \sum_{N=0}^{\infty} a_N z^N \\ &= \left\{ b_0 + b_1 z + \ldots + b_{\beta m(\beta m+1)-1} z^{\beta m(\beta m+1)-1} \right\} \left\{ 1 - \left(\frac{z}{\rho}\right)^{\beta m(\beta m-1)} \right\}^{-1}. \end{aligned}$$

Clearly $f \in A_\rho$ and that

$$a_N = b_k \rho^{-\beta m(\beta m+1)\nu} \tag{2.5.3}$$

where $N = \beta m(\beta m+1)\nu + k$, $k = 0, 1, \ldots, \beta m(\beta m+1) - 1, \nu = 0, 1, \ldots$.
From (2.5.2) and (2.5.3), we have

$$\left(\sum_{k=0}^{\beta m-1} a_{\beta mn+k} z_j^k \right)^{(p_j-1)} = 0 \qquad \text{for each } n \text{ and } j = 1, 2, \ldots, \mu. \tag{2.5.4}$$

and

$$\left(\sum_{k=0}^{\beta m} a_{(\beta m+1)n+k} z_j^k \right)^{(p_j-1)} = 0 \qquad \text{for each } n \text{ and } j = \mu+1, \ldots, s. \tag{2.5.5}$$

For any integer $n > 0$ let r and t be determined by

$$\beta mn + t = (\beta m+1)r, \qquad 0 \le t < \beta m + 1$$

then for $j \ge \mu + 1$ from (2.5.5)

$$\begin{aligned} \left(\sum_{k=0}^{n-1} a_{k+\beta mn} z_j^k \right)^{(p_j-1)} &= \left(\sum_{k=0}^{t-1} a_{k+\beta mn} z_j^k + \sum_{k=t}^{n-1} a_{k+\beta mn} z_j^k \right)^{(p_j-1)} \\ &= \left(\sum_{k=0}^{t-1} a_{k+\beta mn} z_j^k + \sum_{k=0}^{\beta m} \sum_{\nu=r}^{n-1} a_{(\beta m+1)\nu+k} z_j^{(\beta m+1)\nu+k-\beta mn} \right)^{(p_j-1)} \\ &= \left(\sum_{k=0}^{t-1} a_{k+\beta mn} z_j^k + 0 \right)^{(p_j-1)} \\ &= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{\beta mn}} \right). \qquad \text{(for large } n\text{)} \end{aligned}$$

This for $\mu < j \le s$ gives

$$\begin{aligned}
\Delta^{(p_j-1)}_{n-1,l,m}(z_j;f) &= \left(\sum_{t=\beta}^{\infty}\sum_{k=0}^{n-1} a_{k+qt}z_j^k\right)^{(p_j-1)} \\
&= \left(\sum_{t=\beta}^{\infty}\sum_{k=0}^{n-1} a_{k+tmn}z_j^k\right)^{(p_j-1)} \\
&= \left(\sum_{k=0}^{n-1} a_{k+\beta mn}z_j^k + \sum_{t=\beta+1}^{\infty}\sum_{k=0}^{n-1} a_{k+tmn}z_j^k\right)^{(p_j-1)} \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{\beta mn}}\right) + \mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{n(m(\beta+1)+1)}}\right). \qquad (2.5.6)
\end{aligned}$$

This together with (2.3.8) and (2.3.9) gives

$$\Delta^{(p_j-1)}_{n-1,l,m}(z_j;f) = \mathcal{O}\left(\frac{|z_j|}{\rho^{\beta m+1}} - \eta\right)^n \qquad \text{for} \qquad |z_j| > \rho. \qquad (2.5.7)$$

Now, let for any integer $n > 0$, r and t be determined by

$$\beta mr + t = (\beta m + 1)n, \qquad 0 \le t < \beta m.$$

Then for $0 \le j \le \mu$ from (2.5.4) we have

$$\begin{aligned}
\left(\sum_{k=0}^{n-1} a_{k+\beta mn}z_j^k\right)^{(p_j-1)} &= \left(\sum_{k=\beta mn}^{\beta mn+n-1} a_k z_j^{k-n\beta m}\right)^{(p_j-1)} \\
&= \left(\sum_{k=\beta mn}^{r\beta m-1} a_k z_j^{k-n\beta m} + \sum_{k=r\beta m}^{(\beta m+1)n-1} a_k z_j^{k-n\beta m}\right)^{(p_j-1)} \\
&= \left(\sum_{\nu=n}^{r-1}\sum_{k=0}^{\beta m-1} a_{k+\beta m\nu} z_j^{k+\beta m(\nu-n)} + \sum_{k=r\beta m}^{(\beta m+1)n-1} a_k z_j^{k-\beta mn}\right)^{(p_j-1)} \\
&= \left(\sum_{k=0}^{t-1} a_{k+r\beta m} z_j^{k+\beta m(r-n)}\right)^{(p_j-1)} \\
&= \mathcal{O}\left(\frac{|z_j|^{\beta m(r-n)}}{(\rho-\epsilon)^{r\beta m}}\right) \\
&= \mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{(\beta m+1)n}}\right)
\end{aligned}$$

whence for $0 \le j \le \mu$ we have

$$\begin{aligned}
\Delta^{(p_j-1)}_{n-1,1,q}(z_j;f) &= \left(\sum_{k=0}^{n-1} a_{k+\beta mn}z_j^k + \sum_{t=\beta+1}^{\infty}\sum_{k=0}^{n-1} a_{k+tmn}z_j^k\right)^{(p_j-1)} \\
&= \mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{(\beta m+1)n}} + \frac{1}{(\rho-\epsilon)^{m(\beta+1)n}}\right). \ |z_j| < \rho \qquad (2.5.8)
\end{aligned}$$

This togeher with (2.3.21) and (2.3.22) gives

$$\Delta^{(p_j-1)}_{n-1,l,m}(z_j;f)=\mathcal{O}\left(\frac{1}{\rho^{\beta m}}-\eta\right)^n \qquad \text{for} \qquad |z_j|<\rho. \tag{2.5.9}$$

Hence (2.5.7) and (2.5.9) gives

$$H^{(p_j-1)}_{\beta,m}(z_j;f)<K_{\beta,m}(|z_j|,\rho)$$

For the converse part suppose $H^{(p_j-1)}_{\beta,m}(z_j;f)<K_{\beta,m}(|z_j|,\rho)$ $(j=1,2,\ldots,s)$ for some $f=\sum_{k=0}^{\infty}a_kz^k\in A_\rho$ and that $rank\ M=\beta m(\beta m+1)$. Set

$$\begin{aligned}
h^{(p_j-1)}(z) &= \Delta^{(p_j-1)}_{n-1,l,m}(z;f)-z^{\beta m}\Delta^{(p_j-1)}_{n,l,m}(z;f)\\
&= \left(\sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1}a_{jmn+k}z^k-z^{\beta m}\sum_{j=\beta}^{\infty}\sum_{k=0}^{n}a_{jm(n+1)+k}z^k\right)^{(p_j-1)}\\
&= \left(\sum_{k=0}^{n-1}a_{\beta mn)+k}z^k-\sum_{k=0}^{n}a_{\beta m(n+1)+k}z^{k+\beta m}+\right.\\
&\quad \left.+\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n}a_{jmn+k}z^k-\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n}a_{jm(n+1)+k}z^{k+\beta m}\right)^{(p_j-1)}\\
&= \left(\sum_{k=0}^{n-1}a_{\beta mn+k}z^k-\sum_{k=\beta m}^{n+\beta m}a_{\beta mn+k}z^k+\right.\\
&\quad \left.+\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1}a_{jmn+k}z^k-\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n}a_{jm(n+1)+k}z^{k+\beta m}\right)^{(p_j-1)}\\
&= \left(\left(\sum_{k=0}^{\beta m-1}+\sum_{k=\beta m}^{n-1}-\sum_{k=\beta m}^{n-1}-\sum_{k=n}^{n+\beta m}\right)a_{\beta mn+k}z^k\right)^{(p_j-1)}+\mathcal{O}((K_{\beta+1,m}(|z|,\rho-\epsilon))^n)\\
&= \left(\sum_{k=0}^{\beta m-1}a_{k+\beta mn}z^k-\sum_{k=0}^{\beta m}a_{k+\beta mn+n}z^{k+n}\right)^{(p_j-1)}+\mathcal{O}((K_{\beta+1,m}(|z|,\rho-\epsilon))^n).
\end{aligned}\tag{2.5.10}$$

Now for $0\le j\le\mu$ from (2.3.21) and (2.3.22)

$$\begin{aligned}
h^{(p_j-1)}(z_j) &= \left(\sum_{k=0}^{\beta m-1}a_{k+\beta mn}z_j^k\right)^{(p_j-1)}+\\
&\quad +\mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{(\beta m+1)n}}+\frac{1}{(\rho-\epsilon)^{(\beta+1)mn}}\right)\\
&= \left(\sum_{k=0}^{\beta m-1}a_{k+\beta mn}z_j^k\right)^{(p_j-1)}+\mathcal{O}\left(\frac{1}{\rho^{\beta m}}-\eta\right)^n.
\end{aligned}\tag{2.5.11}$$

Now from hypothesis $H^{(p_j-1)}_{\beta,m}(z_j;f)<K_{\beta,m}(|z_j|,\rho)$ $\qquad (j=1,2,\ldots,\mu)$. That is

$$\overline{\lim_{n\to\infty}}|\Delta^{(p_j-1)}_{n-1,l,m}(z;f)|^{1/n}=\frac{1}{\rho^{\beta m}}-\eta$$

for some $\eta > 0$. Thus,

$$\Delta_{n-1,l,m}^{(p_j-1)}(z_j; f) \leq \left(\frac{1}{\rho^{\beta m}} - \eta + \epsilon\right)^n$$

for $n \geq n_0(\epsilon)$ and $\eta > \epsilon > 0$. Thus

$$\begin{aligned} h^{(p_j-1)}(z_j) &= \Delta_{n-1,l,m}^{(p_j-1)}(z_j; f) - z_j^{\beta m} \Delta_{n,l,m}^{(p_j-1)}(z_j; f) \\ &= \mathcal{O}\left(\frac{1}{\rho^{\beta m}} - \eta\right)^n. \end{aligned}$$

Hence from (2.5.11) we obtain

$$\left(\sum_{k=0}^{\beta m-1} a_{k+\beta mn} z_j^k\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{\beta m}} - \eta\right)^n. \tag{2.5.12}$$

Similarly for $j > \mu$ from (2.5.10) from (2.3.8) and (2.3.9) we have

$$\begin{aligned} h^{(p_j-1)}(z_j) &= \left(-\sum_{k=0}^{\beta m} a_{k+\beta mn+n} z_j^{k+n}\right)^{(p_j-1)} + \\ &\quad + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{\beta mn}} + \frac{|z_j|^n}{(\rho-\epsilon)^{((\beta+1)m+1)n}}\right) \\ &= \left(-\sum_{k=0}^{\beta m} a_{k+\beta mn+n} z_j^{k+n}\right)^{(p_j-1)} + \mathcal{O}\left(\frac{|z_j|}{\rho^{\beta m+1}} - \eta\right)^n. \end{aligned} \tag{2.5.13}$$

Now from hypothesis $H_{\beta,m}^{(p_j-1)}(z_j; f) < K_{\beta,m}(|z_j|, \rho)$ $(j = \mu+1, \ldots, s)$. That is

$$\overline{\lim_{n\to\infty}} |\Delta_{n-1,l,m}^{(p_j-1)}(z_j; f)|^{1/n} = \frac{|z_j|}{\rho^{(\beta m+1)}} - \eta$$

for some $\eta > 0$. Thus,

$$\Delta_{n-1,l,m}^{(p_j-1)}(z_j; f) \leq \left(\frac{|z_j|}{\rho^{\beta m+1}} - \eta + \epsilon\right)^n$$

for $n \geq n_0(\epsilon)$ and $\eta > \epsilon > 0$. Thus

$$\begin{aligned} h^{(p_j-1)}(z_j) &= \Delta_{n-1,l,m}^{(p_j-1)}(z_j; f) - z_j^{\beta m} \Delta_{n,l,m}^{(p_j-1)}(z_j; f) \\ &= \mathcal{O}\left(\frac{|z_j|}{\rho^{\beta m+1}} - \eta\right)^n. \end{aligned}$$

Hence from (2.5.13) we obtain

$$\left(\sum_{k=0}^{\beta m} a_{k+\beta mn+n} z_j^{k+n}\right)^{(p_j-1)} = \mathcal{O}\left(\frac{|z_j|}{\rho^{\beta m+1}} - \eta\right)^n$$

or,

$$\left(\sum_{k=0}^{\beta m} a_{k+\beta mn+n} z_j^{k}\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{\beta m+1}} - \eta_1\right)^n. \tag{2.5.14}$$

Now, since (2.5.12) and (2.5.14) hold for all n, putting $n = (\beta m+1)\nu+\lambda, \lambda = 0,\ldots,\beta m$ in (2.5.12) and $n = \beta m\nu+\lambda, \lambda = 0,\ldots,\beta m-1$ in (2.5.14) we have

$$\left(\sum_{k=0}^{\beta m-1} a_{k+\beta m(\beta m+1)\nu+\lambda\beta m} z_j^k\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{\beta m}}-\eta\right)^{(\beta m+1)\nu+\lambda} \tag{2.5.15}$$

$(j = 1,\ldots,\mu; \lambda = 0,1,\ldots \beta m; \nu = 0,1,\ldots)$,

$$\left(\sum_{k=0}^{\beta m} a_{k+(\beta m+1)\beta m\nu+\lambda(\beta m+1)} z_j^k\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{\beta m+1}}-\eta\right)^{\beta m\nu+\lambda} \tag{2.5.16}$$

$(j = \mu+1,\ldots,s; \lambda = 0,1,\ldots \beta m-1; \nu = 0,1,\ldots)$.

Now from (2.3.30)

$$\left(\frac{1}{\rho^{\beta m}}-\eta\right)^{(\beta m+1)\nu} < \left(\frac{1}{\rho^{\beta m(\beta m+1)}}-\eta_1\right)^{\nu}.$$

Hence (2.5.15) can be written as

$$\left(\sum_{k=0}^{\beta m-1} a_{k+\beta m(\beta m+1)\nu+\lambda\beta m} z_j^k\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{\beta m(\beta m+1)}}-\eta\right)^{\nu} \tag{2.5.17}$$

$(j = 1,\ldots,\mu; \lambda = 0,1,\ldots \beta m; \nu = 0,1,\ldots)$.

Similarly (2.5.16) can be written as

$$\left(\sum_{k=0}^{\beta m} a_{k+(\beta m+1)\beta m\nu+\lambda(\beta m+1)} z_j^k\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{\beta m(\beta m+1)}}-\eta\right)^{\nu} \tag{2.5.18}$$

$(j = \mu+1,\ldots,s; \lambda = 0,1,\ldots \beta m-1; \nu = 0,1,\ldots)$.

Note that (2.5.17) and (2.5.18) can be written as

$$M.A^T = B \tag{2.5.19}$$

where

$$A = \left(a_{\beta m(\beta m+1)\nu}, a_{\beta m(\beta m+1)\nu+1}, \ldots, a_{\beta m(\beta m+1)\nu+\beta m(\beta m+1)-1}\right)$$

and

$$B = \left(\mathcal{O}\left(\frac{1}{\rho^{\beta m(\beta m+1)}}-\eta\right)^{\nu}\right),$$

B is a column vector of order $((s\beta m+\mu)\times 1)$.

Since $rank\ \ M = \beta m(\beta m+1)$, solving (2.5.19) we get

$$a_{\beta m(\beta m+1)\nu+k} = \mathcal{O}\left(\frac{1}{\rho^{\beta m(\beta m+1)}}-\eta\right)^{\nu}$$

for $k = 0, 1, \ldots, \beta m(\beta m + 1) - 1$. Hence

$$\overline{\lim_{\nu \to \infty}} |a_\nu|^{1/\nu} < \frac{1}{\rho}$$

which is a contradiction to $f \in A_\rho$.

Remark 2.5.1 For $m = 1$ Theorem 2.5.1 gives Theorem 2.1.9.

Remark 2.5.2 For $p_\nu = 1 \forall \nu$ Theorem 2.5.1 gives Theorem 2.3.1.

Chapter 3

WALSH OVERCONVERGENCE USING LEAST SQUARE APPROXIMATING POLYNOMIALS

3.1 Rivlin [39], by considering least square approximation to functions in A_ρ (class of functions analytic in $|z| < \rho$ but not in $|z| \le \rho$) by polynomials of degree n on the q^{th} roots of unity ($q \ge n+1$), generalized the Walsh equiconvergence theorem in the following manner :

Theorem 3.1.1 [39] *If $q = mn + c, m \ge 1, 0 \le c < m$ and $P_n(z)$ denotes the polynomial of degree n which minimizes*

$$\sum_{k=0}^{q-1} |Q_n(\omega^k) - f(\omega^k)|^2, \qquad \omega^q = 1$$

over all polynomials Q_n of degree $\le n$ and $S_n(z)$ is Taylor polynomial of degree n for $f \in A_\rho$, then

$$\lim_{n\to\infty} \{P_n(z) - S_n(z)\} = 0, \qquad \forall |z| < \rho^{1+m}.$$

Recently, the above result of Rivlin has been extended by Cavaretta, Dikshit and Sharma [10] as follows :

Let $q = mn + c, 0 \le c < m, n \ge 0$ and $P_{n-1,r}(z; f)$ be the polynomial which minimizes

$$\sum_{\nu=0}^{r-1} \sum_{k=0}^{q-1} |Q_{n-1}^{(\nu)}(\omega^k) - f^{(\nu)}(\omega^k)|^2, \;\; \omega^q = 1, r \;\; \text{a fixed integer}$$

over all polnomials Q_{n-1} of degree $\le n-1$. Then from Lemma 2 [10] if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in$

A_ρ, then $P_{n-1,r}$ is given by

$$P_{n-1,r}(z;f) = \sum_{k=0}^{n-1} c_k z^k$$

where

$$c_k = \frac{1}{A_{0,k}(r)} \sum_{j=0}^{\infty} A_{j,k}(r) a_{k+qj}, \qquad k = 0, 1, \ldots, n-1$$

and

$$A_{j,k}(r) = \sum_{i=0}^{r-1} (k)_i (k+jq)_i,$$

where $(k)_i = k(k-1), \ldots, (k-i+1)$ and $(k)_0 = 1$.

Set

$$S_{n-1,j,r}(z;f) = \sum_{k=0}^{n-1} \frac{A_{j,k}(r)}{A_{0,k}(r)} a_{k+qj} z^k, \qquad j = 0, 1, \ldots$$

and for a fixed positive integer l,

$$\Delta_{n-1,l,q}(z;f) = P_{n-1,r}(z;f) - \sum_{j=0}^{l-1} S_{n-1,j,r}(z;f)$$

then we have

Theorem 3.1.2 [10] *For any $f \in A_\rho, \rho > 1$ and for any positive integer l we have, for $R \geq \rho$,*

$$\lim_{n\to\infty} \Delta_{n-1,l,q}(z;f) = 0, \qquad \forall |z| < \rho^{1+ml}.$$

Further L. Yuanren [25] and M.P.Stojanova [50] generalised Walsh's theorem by considering $D_\rho = \{z \in C; |z| < \rho\}, \Gamma_\rho = \{z \in C; |z| = \rho\}$. That is A_ρ denote the set of all functions $f(z)$ which are analytic in D_ρ but not on Γ_ρ. Let $\alpha, \beta \in D_\rho$ and for any positive integer s and $n(s > n)$ let $L_{n-1}(z, \alpha, f)$ and $L_{s-1}(z, \beta, f)$ denote the Lagrange interpolants of f in the zeros of $z^n - \alpha^n$ and $z^s - \beta^s$ respectively. With above notations L. Yuanren [25] proved

Theorem 3.1.3 [25] *If $s = s_n = ln + p, p = p_n = r_1 n + \mathcal{O}(1), 0 \leq r_1 < 1, p \geq 0$ then for each $f \in A_\rho$ and for each $\alpha, \beta \in D_\rho$, we have*

$$\overline{\lim_{n\to\infty}} |\Delta_{n,s}^{\alpha,\beta}(z;f)|^{1/n} = 0, \qquad \forall\ |z| < \tau,$$

where

$$\Delta_{n,s}^{\alpha,\beta}(z;f) = L_{n-1}(z,\alpha,f) - L_{n-1}(z,\alpha,L_{s-1}(z,\beta,f))$$

and

$$\tau = \rho/max\{|\alpha/\rho|^l, |\beta/\rho|^{l+r_1}\}.$$

More precisely for any R with $\rho < R < \infty$, we have

$$\overline{\lim_{n\to\infty}}\{\max_{z\in D_R}|\Delta_{n,s}^{\alpha,\beta}(z;f)|^{1/n}\} \le R/\tau.$$

When $\alpha = 1, \beta = 0$ and $s = ln$, the above result yields a result of Cavaretta et al [12], which itself is a generalisation of a theorem of Walsh [58,p.153].

M.P.Stojanova [50] obtained more precise theorem for the difference $\Delta_{n,s}^{\alpha,\beta}$:

Theorem 3.1.4 [50] *With the hypothesis of Theorem 3.1.3, if $|\alpha/\rho|^l \ne |\beta/\rho|^{l+r_1}$ and for $r_1 \ne 0$ if $|\alpha/\rho|^{l+1} \ne |\beta/\rho|^{l+r_1}$, then*

$$\overline{\lim_{n\to\infty}}\{\max_{|z|=R}|\Delta_{n,s}^{\alpha,\beta}(z;f)|^{1/n}\} = K_\rho(R), \qquad R > 0,$$

where

$$K_\rho(R) = \begin{cases} (R/\rho)max\{|\alpha/\rho|^l, |\beta/\rho|^{l+r_1}\} & for\;\; R \ge \rho \\ max\{|\alpha/\rho|^{l+1}, |\alpha/\rho|^l(R/\rho)^{r_1}, |\beta/\rho|^{l+r_1}\} & for\;\; 0 < R < \rho \end{cases}$$

As a particular case $\alpha = 1, \beta = 0$ and $s_n = ln$, Theorem 3.1.4 reduces to Theorem 2.1.2.

Now if $S_{n-1}(z;g)$ denotes the $(n-1)^{th}$ partial sum of power series of g then

$$S_{lq-1}(z;f) = \sum_{k=0}^{lq-1} a_k z^k = \sum_{k=0}^{q-1}\sum_{j=0}^{l-1} a_{k+qj} z^{k+qj}.$$

Since for

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{q-1}\sum_{j=0}^{\infty} a_{k+qj} z^{k+qj},$$

we have

$$P_{n-1,r}(z;f) = \sum_{k=0}^{n-1}\sum_{j=0}^{\infty} \frac{A_{j,k}(r)}{A_{0,k}(r)} a_{k+qj} z^k.$$

Hence

$$P_{n-1,r}[z;S_{lq-1}(z;f)] = \sum_{k=0}^{n-1}\sum_{j=0}^{l-1} \frac{A_{j,k}(r)}{A_{0,k}(r)} a_{k+qj} z^k. \tag{3.1.1}$$

Thus motivated by Theorem 3.1.3, Theorem 3.1.2 can be stated as

Theorem 3.1.5 *For each $f \in A_\rho$ and each positive integer l,*

$$\lim_{n\to\infty}\{P_{n-1,r}(z;f) - P_{n-1,r}[z;S_{lq-1}(z;f)]\} = 0, \qquad for \qquad |z| < \rho^{1+lm}. \tag{3.1.2}$$

the convergence being uniform and geometric on any closed subset of $z \leq Z < \rho^{1+lm}$. *Moreover the result of (3.1.2) is best possible.*

In the present chapter in section (3.2) motivated by Totik [56] we extend and improve the above Theorem 3.1.2 by obtaining a result that gives exact estimates for the growth of $\Delta_{n-1,l,q}(z, f)$ not only for $R \geq \rho$ but for all positive R. In section (3.3) motivated by M.P.Stojanova [50] we consider roots of α^n and extend Theorem 3.2.1. Theorem 3.2.1 has been published in the Journal of Analysis [24].

3.2 If we set

$$g_{l,m}(R) = \overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{n-1,l,q}(z; f)|^{1/n},$$

and

$$K_{l,m}(R, \rho) = \begin{cases} \frac{R}{\rho^{1+ml}}, & \text{if} \quad R \geq \rho \\ \frac{1}{\rho^{ml}} & \text{if} \quad 0 < R < \rho. \end{cases}$$

Then, from Theorem 3.1.2 for $R > \rho$

$$g_{l,m}(R) \leq K_{l,m}(R, \rho).$$

We shall prove that here equality holds. We also consider the pointwise behaviour of $\Delta_{n-1,l,q}(z; f)$.

Theorem 3.2.1 *If* $f \in A_\rho, l$ *is a positive integer and* $R > 0$ *then*

$$g_{l,m}(R) = K_{l,m}(R, \rho)$$

Before proving the theorem we give a lemma.

Lemma 3.2.1 *For* $|t| > 1$ *and* $i \geq 0, q \geq 1, k \geq 0$

$$S_{q,l}(t) = (-1)^i t \frac{d^i}{dt^i} \left(\frac{t^{(1-l)q+i-k-1}}{t^q - 1} \right) = \sum_{j=l}^{\infty} \frac{(k + jq)_i}{t^{k+jq}}.$$

Further,

$$S_{q,l}(t) = \mathcal{O}\left((k + lq)_i |t|^{-lq-k}\right),$$

for large n *that is for large* q.

Proof:

$$S_{q,l}(t) \;=\; (-1)^i t \frac{d^i}{dt^i} \left(\frac{t^{(1-l)q+i-k-1}}{t^q - 1} \right)$$

$$
\begin{aligned}
&= (-1)^i t \frac{d^i}{dt^i}\left(\frac{t^{(1-l)q+i-k-1}}{t^q}\sum_{j=0}^{\infty} t^{-jq}\right) \\
&= (-1)^i t \frac{d^i}{dt^i}\left(t^{i-k-1}\sum_{j=l}^{\infty} t^{-jq}\right) \\
&= (-1)^i t \sum_{j=l}^{\infty} \frac{d^i}{dt^i}\left(t^{i-k-1-jq}\right) \\
&= (-1)^i t \sum_{j=l}^{\infty}(i-k-1-jq)(i-k-1-jq-1)\ldots \\
&\quad (i-k-1-jq-(i-1))t^{i-k-1-jq-i} \\
&= \sum_{j=l}^{\infty}(k+jq)(k+jq-1)\ldots(k+jq-i+1)t^{-k-jq} \\
&= \sum_{j=l}^{\infty}\frac{(k+jq)_i}{t^{k+jq}}.
\end{aligned}
$$

Further

$$
\begin{aligned}
S_{q,l}(t) &= (-1)^i t \frac{d^i}{dt^i}\left(\frac{t^{(1-l)q+i-k-1}}{t^q-1}\right) \\
&= \mathcal{O}\left(\frac{d^i}{dt^i}\left(\frac{t^{-lq+i-k-1}}{1-t^{-q}}\right)\right) \\
&= \mathcal{O}\left((k+lq)_i|t|^{-lq-k}\right).
\end{aligned}
$$

Proof of Theorem 3.2.1 : Since $f \in A_\rho$, we have

$$a_k = \mathcal{O}(\rho-\epsilon)^{-k} \tag{3.2.1}$$

for every ϵ satisfying $0 < \epsilon < \rho - 1$ and $k \geq k_0(\epsilon)$. Let R be fixed, $\mid z \mid = R$ and if $R < \rho$ then we assume $\epsilon > 0$ so small that $R < \rho - \epsilon$ be satisfied as well. Then by the definition of $\Delta_{n-1,l,q}(z; f)$ and above Lemma 3.2.1 we obtain

$$
\begin{aligned}
\Delta_{n-1,l,q}(z;f) &= \sum_{j=l}^{\infty}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}z^k \\
&= \mathcal{O}\left(\sum_{j=l}^{\infty}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}}\frac{\mid z\mid^k}{(\rho-\epsilon)^{k+qj}}\right) \\
&= \mathcal{O}\left(\sum_{j=l}^{\infty}\sum_{k=0}^{n-1}\frac{\sum_{i=0}^{r-1}(k)_i(k+jq)_i}{\sum_{i=0}^{r-1}(k)_i(k)_i}\frac{\mid z\mid^k}{(\rho-\epsilon)^{k+qj}}\right) \\
&= \mathcal{O}\left(\sum_{k=0}^{n-1}\sum_{i=0}^{r-1}(k)_i|z|^k\sum_{j=l}^{\infty}\frac{(k+jq)_i}{(\rho-\epsilon)^{k+qj}}\right) \\
&= \mathcal{O}\left(\sum_{k=0}^{n-1}\sum_{i=0}^{r-1}(k)_i|z|^k S_{q,l}(\rho-\epsilon)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{O}\left(\sum_{k=0}^{n-1}\sum_{\imath=0}^{r-1}(k)_\imath |z|^k (k+lq)_\imath (\rho-\epsilon)^{-lq-k}\right) \\
&= \mathcal{O}\left((\rho-\epsilon)^{-lq}\sum_{k=0}^{n-1}\sum_{\imath=0}^{r-1}(k)_\imath (k+lq)_\imath \frac{R^k}{(\rho-\epsilon)^{-k}}\right) \\
&= \mathcal{O}\begin{cases} N(n)\frac{R^n}{(\rho-\epsilon)^{lq+n}} & \text{for} \quad R \geq \rho \\ N(n)\frac{1}{(\rho-\epsilon)^{lq}} & \text{for} \quad 0 < R < \rho, \end{cases}
\end{aligned}
$$

where $N(n)$ is a quantity dependent on n with $\lim_{n\to\infty}(N(n))^{1/n} = 1$. Thus

$$
\begin{aligned}
\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{n-1,l,q}(z;f)|^{1/n} &\leq \frac{R}{(\rho-\epsilon)^{1+ml}}, \quad \text{if} \quad R \geq \rho \\
&\leq \frac{1}{(\rho-\epsilon)^{ml}} \quad \text{if} \quad 0 < R < \rho.
\end{aligned}
$$

Being $\epsilon > 0$ arbitrary small this gives

$$
\begin{aligned}
\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{n-1,l,q}(z;f)|^{1/n} &\leq \frac{R}{\rho^{1+ml}}, \quad \text{if} \quad R \geq \rho \\
&\leq \frac{1}{\rho^{ml}} \quad \text{if} \quad 0 < R < \rho.
\end{aligned}
$$

To prove the opposite inequality let first $R \geq \rho$, then

$$
\begin{aligned}
\Delta_{n-1,l,q}(z;f) &= \sum_{\jmath=l}^{\infty}\sum_{k=0}^{n-1}\frac{A_{\jmath,k}}{A_{0,k}}a_{k+q\jmath}z^k \\
&= \sum_{k=0}^{n-ml-2}\frac{A_{l,k}}{A_{0,k}}a_{k+ql}z^k + \sum_{k=n-ml-1}^{n-1}\frac{A_{l,k}}{A_{0,k}}a_{k+ql}z^k + \\
&\quad + \sum_{\jmath=l+1}^{\infty}\sum_{k=0}^{n-1}\frac{A_{\jmath,k}}{A_{0,k}}a_{k+q\jmath}z^k.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{k=n-ml-1}^{n-1}\frac{A_{l,k}}{A_{0,k}}a_{k+ql}z^k &= \Delta_{n-1,l,q}(z;f) - \sum_{k=0}^{n-ml-2}\frac{A_{l,k}}{A_{0,k}}a_{k+ql}z^k - \\
&\quad - \sum_{\jmath=l+1}^{\infty}\sum_{k=0}^{n-1}\frac{A_{\jmath,k}}{A_{0,k}}a_{k+q\jmath}z^k
\end{aligned}
$$

gives, by Cauchy integral formula, for $n - ml - 1 \leq k \leq n-1$,

$$
\begin{aligned}
\frac{A_{l,k}}{A_{0,k}}a_{k+ql} &= \frac{1}{2\pi i}\int_{|z|=R}\frac{\Delta_{n-1,l,q}(z;f)}{z^{k+1}}dz - \\
&\quad \frac{1}{2\pi i}\sum_{k'=0}^{n-ml-2}\frac{A_{l,k'}}{A_{0,k'}}a_{k'+ql}\int_{|z|=R}\frac{z^{k'}}{z^{k+1}}dz \\
&\quad -\frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{\jmath=l+1}^{\infty}\sum_{k'=0}^{n-1}\frac{A_{\jmath,k'}}{A_{0,k'}}a_{k'+q\jmath}z^{k'}}{z^{k+1}}dz.
\end{aligned}
$$

Since $\int_{|z|=R} \frac{z^{k'}}{z^{k+1}} dz$ is non zero only for $k = k'$, the middle integral on the right hand side in above equation is zero. By the definition of $g_{l,m}(R)$ and (3.2.1) we have for every $n \geq n_0(\epsilon)$ and a constant M, which need not be same at each occurrence

$$\begin{aligned} \left| \frac{A_{l,k}}{A_{0,k}} a_{k+ql} \right| &\leq M \frac{(g_{l,m}(R)+\epsilon)^n}{R^k} + \mathcal{O}\left(N(n) \frac{R^n}{R^k(\rho-\epsilon)^{n+q(l+1)}} \right) \\ &\leq M \frac{(g_{l,m}(R)+\epsilon)^n}{R^k} + \mathcal{O}\left(N(n) \frac{1}{(\rho-\epsilon)^{n(1+m(l+1))}} \right). \end{aligned}$$

Let $\epsilon > 0$ be so small that

$$(\rho-\epsilon)^{-(1+m(l+1))} < \rho^{-(1+ml)}.$$

Thus,

$$(g_{l,m}(R)+\epsilon)^n \geq \frac{R^k}{M} \left(\left| \frac{A_{l,k}}{A_{0,k}} a_{k+ql} \right| - \mathcal{O}\left(\frac{N(n)}{\rho^{n(1+ml)}} \right) \right)$$

hence,

$$g_{l,m}(R) + \epsilon \geq \overline{\lim_{n\to\infty}} \left\{ |a_{k+ql}|^{\frac{1}{k+ql}} \right\}^{\frac{k+ql}{n}} \left\{ \frac{A_{l,k}}{A_{0,k}} \frac{R^k}{M} \right\}^{\frac{1}{n}}.$$

Now since $n - ml - 1 \leq k \leq n-1$ we have, $\lim_{n\to\infty} \frac{k}{n} = 1$ and so

$$g_{l,m}(R) + \epsilon \geq \frac{R}{\rho^{1+ml}}.$$

Since ϵ is arbitrary, this yeilds

$$g_{l,m}(R) \geq \frac{R}{\rho^{1+ml}} \qquad \text{for} \qquad R \geq \rho.$$

For the case $0 < R < \rho$, we write

$$\begin{aligned} \Delta_{n-1,l,q}(z;f) &= \sum_{j=l}^{\infty} \sum_{k=0}^{n-1} \frac{A_{j,k}}{A_{0,k}} a_{k+qj} z^k \\ &= \sum_{k=0}^{ml-1} \frac{A_{l,k}}{A_{0,k}} a_{k+ql} z^k + \sum_{k=ml}^{n-1} \frac{A_{l,k}}{A_{0,k}} a_{k+ql} z^k + \sum_{j=l+1}^{\infty} \sum_{k=0}^{n-1} \frac{A_{j,k}}{A_{0,k}} a_{k+qj} z^k \end{aligned}$$

whence,

$$\begin{aligned} \sum_{k=0}^{ml-1} \frac{A_{l,k}}{A_{0,k}} a_{k+ql} z^k &= \Delta_{n-1,l,q}(z;f) - \sum_{k=ml}^{n-1} \frac{A_{l,k}}{A_{0,k}} a_{k+ql} z^k - \\ &\quad - \sum_{j=l+1}^{\infty} \sum_{k=0}^{n-1} \frac{A_{j,k}}{A_{0,k}} a_{k+qj} z^k. \end{aligned}$$

By Cauchy integral formula we have, for $0 \leq k \leq ml-1$,

$$\frac{A_{l,k}}{A_{0,k}} a_{k+ql} = \frac{1}{2\pi i} \int_{|z|=R} \frac{\Delta_{n-1,l,q}(z;f)}{z^{k+1}} dz -$$

$$-\frac{1}{2\pi i}\sum_{k'=ml}^{n-1}\frac{A_{l,k'}}{A_{0,k'}}a_{k'+ql}\int_{|z|=R}\frac{z^{k'}}{z^{k+1}}dz$$
$$-\frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{j=l+1}^{\infty}\sum_{k'=0}^{n-1}\frac{A_{j,k'}}{A_{0,k'}}a_{k'+qj}z^{k'}}{z^{k+1}}dz.$$

Using the same arguments as earlier, we then have,

$$\begin{aligned} \left|\frac{A_{l,k}}{A_{0,k}}a_{k+ql}\right| &\le M\frac{(g_{l,m}(R)+\epsilon)^n}{R^k}+\mathcal{O}\left(N(n)\frac{1}{R^k(\rho-\epsilon)^{q(l+1)}}\right) \\ &\le M(g_{l,m}(R)+\epsilon)^n+\mathcal{O}\left(N(n)\frac{1}{(\rho-\epsilon)^{mn(l+1)}}\right) \end{aligned}$$

Let $\epsilon > 0$ be so small that

$$(\rho-\epsilon)^{-(l+1)} < \rho^{-l},$$

then,

$$(g_{l,m}(R)+\epsilon)^n \ge \left|\frac{A_{l,k}}{A_{0,k}}a_{k+ql}\right| - \mathcal{O}\left(N(n)\frac{1}{\rho^{nml}}\right)$$

or,

$$\begin{aligned} g_{l,m}(R)+\epsilon &\ge \varlimsup_{n\to\infty}\left\{|a_{k+ql}|^{\frac{1}{k+ql}}\right\}^{\frac{k+ql}{n}}\left\{\left|\frac{A_{l,k}}{A_{0,k}}\right|\right\}^{\frac{1}{n}} \\ &= \frac{1}{\rho^{ml}}. \end{aligned}$$

Since ϵ is arbitrary, this gives

$$g_{l,m}(R) \ge \frac{1}{\rho^{ml}} \qquad \text{for} \qquad 0 < R < \rho$$

which completes the proof.

Since

$$\Delta_{n-1,l,q}(z;f) = \sum_{j=l}^{\infty}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}z^k,$$

for $R = 0$, that is $z = 0$ we have

$$\Delta_{n-1,l,q}(z;f) = \sum_{j=l}^{\infty}\frac{A_{j,0}}{A_{0,0}}a_{qj}.$$

Now $(k)_0 = 1$, thus by the definition of $A_{j,k}$, $A_{j,0} = 1$ and $A_{0,0} = 1$. Thus,

$$\Delta_{n-1,l,q}(0;f) = \sum_{j=l}^{\infty}a_{qj}.$$

Consider the function

$$\begin{aligned} F(z) &= \frac{z^{c(l+1)}}{\rho^{c(l+1)}}\frac{1}{1-(z/\rho)^{(l+1)m}} \\ &= \sum_{n=0}^{\infty}\left(\frac{z}{\rho}\right)^{(l+1)mn+c(l+1)}. \end{aligned}$$

Note that for $F(z)$, $a_{ql} = a_{lmn+cl} = 0$. Which gives

$$\begin{aligned}\Delta_{n-1,l,q}(0;F) &= \sum_{j=l+1}^{\infty} a_{qj} \\ &= \mathcal{O}\left(\frac{1}{\rho^{(l+1)mn+c(l+1)}}\right).\end{aligned}$$

Hence for $R = 0$

$$g_{l,m}(R) \leq \frac{1}{\rho^{(l+1)m}} < \frac{1}{\rho^{lm}}.$$

Whence

Remark 3.2.1 For $R = 0$ Theorem 3.2.1 does not hold.

For $r = 1$

$$\begin{aligned}P_{n-1,r}(z;f) &= P_{n-1,1}(z;f) \\ &= \sum_{j=l}^{\infty}\sum_{k=0}^{n-1} a_{k+qj} z^k.\end{aligned}$$

Hence

Remark 3.2.2 For $r = 1$ Theorem 3.2.1 reduces to Theorem 5 [20].

Corollary 3.2.1 *If $l \geq 1$, f is analytic in an open domain containing $|z| \leq 1$ and $g_{l,m}(R) = K_{l,m}(R,\rho)$ for some $R > 0, \rho > 1$ then $f \in A_\rho$.*

Proof Given that f is analytic in an open domain containing $|z| \leq 1$. Hence $f \in A_{\rho'}$ for some $\rho' > 1$. Thus by Theorem 3.2.1 $g_{l,m}(R) = K_{l,m}(R,\rho')$, and from the hypothesis $g_{l,m}(R) = K_{l,m}(R,\rho)$. That is $K_{l,m}(R,\rho') = K_{l,m}(R,\rho)$ and hence $\rho' = \rho$ which gives $f \in A_\rho$.

Remark 3.2.3 For $r = 1, c = 0$ and $m = 1$ Theorem 3.2.1 reduces to Theorem 2.1.2.

Next we consider the pointwise behavior of $\Delta_{n-1,l,q}(z,f)$. We shall prove not only that the sequence $\Delta_{n-1,l,q}(z,f)$ is bounded at most at ml points in $|z| > \rho^{1+ml}$ but

Theorem 3.2.2 *Let $f \in A_\rho, \rho > 1$ and $l \geq 1$. Then*

$$(i) \qquad \overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z,f) \mid^{1/n} = \frac{\mid z \mid}{\rho^{1+lm}}$$

for all but at most lm distinct points in $|z| > \rho$.

(ii) $$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z,f) \mid^{1/n} = \frac{1}{\rho^{lm}}$$

for all but at most $lm-1$ *distinct points in* $0 < |z| < \rho$.

Proof : Let first $|z| = R > \rho$. Consider

$$\Theta_{n-1,l,q}(z;f) = \sum_{j=l}^{\infty}\sum_{k=0}^{n-1}\frac{A_{j,k+ml}}{A_{0,k+ml}}a_{k+qj}z^k. \tag{3.2.2}$$

Now since $f \in A_\rho$ and $q = mn + c$ so

$$\sum_{j=l}^{\infty}\frac{A_{j,k+ml}}{A_{0,k+ml}}a_{k+qj} = \mathcal{O}\left(N(n)(\rho-\epsilon)^{-(k+mln)}\right),$$

whence,

$$\begin{aligned}
\overline{\lim_{k\to\infty}} \Big| \sum_{j=l}^{\infty}\frac{A_{j,k+ml}}{A_{0,k+ml}}a_{k+qj}\Big|^{1/k} &\le \overline{\lim_{k\to\infty}}\left(KN(n)(\rho-\epsilon)^{-(k+mln)}\right)^{1/k} \\
&\le (\rho-\epsilon)^{-1} \\
&< 1.
\end{aligned}$$

Thus sequence (3.2.2) is convergent. Also from the expression of $\Delta_{n-1,l,q}(z;f)$ and $\Theta_{n-1,l,q}(z;f)$ it is clear that

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z,f) \mid^{1/n} = \overline{\lim_{n\to\infty}} \mid \Theta_{n-1,l,q}(z,f) \mid^{1/n} . \tag{3.2.3}$$

Thus,

$$\begin{aligned}
h(z) &= \Delta_{n-1,l,q}(z,f) - z^{lm}\Theta_{n,l,q}(z,f) \\
&= \sum_{j=l}^{\infty}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}}a_{k+(nm+c)j}z^k - z^{lm}\sum_{j=l}^{\infty}\sum_{k=0}^{n}\frac{A_{j,k+ml}}{A_{0,k+ml}}a_{k+((n+1)m+c)j}z^k \\
&= \sum_{k=0}^{n-1}\frac{A_{l,k}}{A_{0,k}}a_{k+(nm+c)l}z^k + \sum_{j=l+1}^{\infty}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}}a_{k+(nm+c)j}z^k - \\
&\quad -z^{lm}\sum_{k=0}^{n}\frac{A_{l,k+ml}}{A_{0,k+ml}}a_{k+((n+1)m+c)l}z^k - z^{lm}\sum_{j=l+1}^{\infty}\sum_{k=0}^{n}\frac{A_{j,k+ml}}{A_{0,k+ml}}a_{k+((n+1)m+c)j}z^k \\
&= \sum_{k=0}^{n-1}\frac{A_{l,k}}{A_{0,k}}a_{k+(nm+c)l}z^k - \sum_{k=lm}^{n+lm}\frac{A_{l,k}}{A_{0,k}}a_{k+(nm+c)l}z^k + \\
&\quad + \sum_{j=l+1}^{\infty}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}}a_{k+(nm+c)j}z^k - z^{lm}\sum_{j=l+1}^{\infty}\sum_{k=0}^{n}\frac{A_{j,k+ml}}{A_{0,k+ml}}a_{k+((n+1)m+c)j}z^k
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{lm-1} \frac{A_{l,k}}{A_{0,k}} a_{k+(nm+c)l} z^k + \sum_{k=lm}^{n-1} \frac{A_{l,k}}{A_{0,k}} a_{k+(nm+c)l} z^k - \\
&\quad - \sum_{k=lm}^{n-1} \frac{A_{l,k}}{A_{0,k}} a_{k+(nm+c)l} z^k - \sum_{k=n}^{n+lm} \frac{A_{l,k}}{A_{0,k}} a_{k+(nm+c)l} z^k \\
&\quad + \sum_{j=l+1}^{\infty} \sum_{k=0}^{n-1} \frac{A_{j,k}}{A_{0,k}} a_{k+(nm+c)j} z^k - z^{lm} \sum_{j=l+1}^{\infty} \sum_{k=0}^{n} \frac{A_{j,k+ml}}{A_{0,k+ml}} a_{k+((n+1)m+c)j} z^k \\
&= \sum_{k=0}^{lm-1} \frac{A_{l,k}}{A_{0,k}} a_{k+(nm+c)l} z^k - \sum_{k=0}^{lm} \frac{A_{l,k+n}}{A_{0,k+n}} a_{k+n(1+lm)+cl} z^{k+n} + \\
&\quad + \sum_{j=l+1}^{\infty} \sum_{k=0}^{n-1} \frac{A_{j,k}}{A_{0,k}} a_{k+(nm+c)j} z^k - z^{lm} \sum_{j=l+1}^{\infty} \sum_{k=0}^{n} \frac{A_{j,k+ml}}{A_{0,k+ml}} a_{k+((n+1)m+c)j} z^k \quad (3.2.4) \\
&= -\sum_{k=0}^{lm} \frac{A_{l,k+n}}{A_{0,k+n}} a_{k+n(1+lm)+cl} z^{k+n} + \mathcal{O}N(n) \left(\frac{|z|^{lm}}{(\rho-\epsilon)^{(n+1)lm}} + \right. \\
&\quad \left. + \frac{|z|^n}{(\rho-\epsilon)^{(1+(l+1)m)n}} \right) + \mathcal{O}\left(\frac{|z|^n}{(\rho-\epsilon)^{n(1+(l+1)m)+(l+1)m}} \right) \\
&= -\sum_{k=0}^{lm} \frac{A_{l,k+n}}{A_{0,k+n}} a_{k+n(1+lm)+cl} z^{k+n} + \mathcal{O}N(n) \left(\frac{1}{(\rho-\epsilon)^{nlm}} + \frac{|z|^n}{(\rho-\epsilon)^{(1+(l+1)m)n}} \right). \\
&\qquad (3.2.5)
\end{aligned}$$

As in (2.3.8) and (2.3.9), by choosing ϵ sufficiently small we can find η a positive number such that

$$\frac{1}{(\rho-\epsilon)^{nlm}} \leq \left(\frac{|z|}{\rho^{lm+1}} - \eta \right)^n \tag{3.2.6}$$

and

$$\frac{|z|^n}{(\rho-\epsilon)^{(1+(l+1)m)n}} \leq \left(\frac{|z|}{\rho^{lm+1}} - \eta \right)^n. \tag{3.2.7}$$

Thus from (3.2.5), (3.2.6) and (3.2.7) we have

$$h(z) = -\sum_{k=0}^{lm} \frac{A_{l,k}}{A_{0,k}} a_{k+n(1+lm)+cl} z^{k+n} + \mathcal{O}N(n) \left(\frac{|z|}{\rho^{1+lm}} - \eta \right)^n \tag{3.2.8}$$

where η is a positive number.

If we assume that in (i) equality does not hold at more than lm points say $lm+1$ points then from Theorem 3.2.1

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z_j, f) \mid^{1/n} < \frac{|z_j|}{\rho^{1+lm}} \qquad , j = 1, 2, \cdots lm+1$$

for $z_1, z_2, \cdots, z_{lm+1}$ with $|z_1|, |z_2|, \cdots, |z_{lm+1}| > \rho$.

Let

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z_j, f) \mid^{1/n} = \frac{|z_j|}{\rho^{1+lm}} - s \qquad \text{for some } s > 0$$

that is

$$|\Delta_{n-1,l,q}(z_j,f)| \leq \left(\frac{|z_j|}{\rho^{1+lm}} - s + \epsilon\right)^n \qquad \forall n > n_0(\epsilon)$$

hence from (3.2.3) we have also that

$$|\Theta_{n-1,l,q}(z_j,f)| \leq \left(\frac{|z_j|}{\rho^{1+lm}} - s + \epsilon\right)^{n+1} \qquad \forall n > n_0(\epsilon)$$

therefore

$$\begin{aligned}
|h(z_j)| &= |\Delta_{n-1,l,q}(z_j,f) - z_j^{lm}\Theta_{n,l,q}(z_j,f)| \\
&\leq |\Delta_{n-1,l,q}(z_j,f)| + |z_j^{lm}\Theta_{n,l,q}(z_j,f)| \\
&\leq \left(\frac{|z_j|}{\rho^{1+lm}} - s + \epsilon\right)^n + |z_j|^{lm}\left(\frac{|z_j|}{\rho^{1+lm}} - s + \epsilon\right)^{n+1} \\
&= \left(\frac{|z_j|}{\rho^{1+lm}} - s + \epsilon\right)^n \left(1 + |z_j|^{lm}\left(\frac{|z_j|}{\rho^{1+lm}} - s + \epsilon\right)\right) \qquad (3.2.9)
\end{aligned}$$

hence

$$\overline{\lim_{n\to\infty}} |h(z_j)|^{1/n} \leq \frac{|z_j|}{\rho^{1+lm}} - s\ , \qquad r > 0$$

that is

$$\overline{\lim_{n\to\infty}} |h(z_j)|^{1/n} < \frac{|z_j|}{\rho^{1+lm}}. \qquad j = 1,2,\cdots,lm+1$$

Now from (3.2.8)

$$\begin{aligned}
\sum_{k=0}^{lm} \frac{A_{l,k+n}}{A_{0,k+n}} a_{k+n(1+lm)+cl} z_j^{k+n} &= \mathcal{O}N(n)\left(\frac{|z_j|}{\rho^{1+lm}} - \eta\right)^n - h(z_j) \\
&= \beta_{j,n}\ (say) \qquad (3.2.10)
\end{aligned}$$

where from (3.2.9) for sufficiently large n and constant $k > 1$

$$\begin{aligned}
|\beta_{j,n}| &\leq \mathcal{O}N(n)\left(\frac{|z_j|}{\rho^{1+lm}} - \eta\right)^n + kN(n)\left(\frac{|z_j|}{\rho^{1+lm}} - s\right)^n \\
&= k_1 N(n)\left(\frac{|z_j|}{\rho^{1+lm}} - \eta_1\right)^n \qquad (3.2.11)
\end{aligned}$$

$$\text{for } k_1 > 1,\ \eta_1 > 0,\ j = 1,2,\cdots,lm+1$$

from (3.2.10)

$$\sum_{k=0}^{lm} \frac{A_{l,k+n}}{A_{0,k+n}} a_{k+n(1+lm)} z_j^k = z_j^{-n}\beta_{j,n} \qquad (3.2.12)$$

where $|\beta_{j,n}| \leq k_1 N(n)\left(\frac{|z_j|}{\rho^{1+lm}} - \eta_1\right)^n$ for suficiently large $n, k_1 > 1, \eta_1 > 0$ and $1 \leq j \leq lm+1$. Solving system of equations (3.2.12) we have. Thus,

$$\frac{A_{l,k+n}}{A_{0,k+n}} a_{(lm+1)n+k+cl} = \sum_{j=1}^{lm+1} c_j^{(k)} z_j^{-n}\beta_{j,n}\ , \quad 0 \leq k \leq lm$$

where matrix $(c_j^{(k)})$, $1 \le j \le lm+1,\ 0 \le k \le lm$ is inverse of coefficint matrix (z_j^k), $1 \le j \le lm+1$, $0 \le k \le lm$ hence $c_j^{(k)}$ are independent of n by which, from (3.2.11) we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \mid a_{(lm+1)n+k+cl} \mid^{1/(lm+1)n+k+cl} \\
&= \overline{\lim_{n\to\infty}} \left(\sum_{j=1}^{lm+1} c_j^{(k)} z_j^{-n} k_1 N(n) \left(\frac{\mid z_j \mid}{\rho^{1+lm}} - \eta_1 \right)^n \right)^{1/(lm+1)n+k+cl} \\
&= \overline{\lim_{n\to\infty}} \left(\sum_{j=1}^{lm+1} c_j^{(k)} k_1 N(n) \left(\frac{1}{\rho^{lm+1}} - \frac{\eta_1}{\mid z_j \mid} \right)^n \right)^{1/(lm+1)n+k+cl} \\
&\le \overline{\lim_{n\to\infty}} \left(k_2 N(n) \left(\frac{1}{\rho^{lm+1}} - \frac{\eta_1}{\mid z_j \mid} \right)^n \right)^{1/(lm+1)n+k+cl} \qquad (3.2.13)
\end{aligned}$$

where $$k_2 = k_1(lm+1) \sum_{j=1}^{lm+1} c_j^{(k)} > 1\,,\ 0 \le k \le lm$$

hence from (3.2.13) we have

$$\overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} < \frac{1}{\rho}$$

which contradicts that $f \in A_\rho$. Hence our assumption that in (i) equality does not hold at more than lm points was wrong and thus

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z,f) \mid^{1/n} = \frac{\mid z \mid}{\rho^{1+lm}}$$

for all but at most lm distinct points in $\mid z \mid > \rho$.

In the proof of (ii), one can argue similarly using (3.2.4)

$$\begin{aligned}
h(z) \;=\; & \sum_{k=0}^{lm-1} \frac{A_{l,k}}{A_{0,k}} a_{k+(nm+c)l} z^k - \sum_{k=0}^{lm} \frac{A_{l,k+n}}{A_{0,k+n}} a_{k+n(1+lm)+cl} z^k + \\
& \sum_{j=l+1}^{\infty} \sum_{k=0}^{n-1} \frac{A_{j,k}}{A_{0,k}} a_{k+(nm+c)j} z^k - z^{lm} \sum_{j=l+1}^{\infty} \sum_{k=0}^{n} \frac{A_{j,k+ml}}{A_{0,k+ml}} a_{k+((n+1)m+c)j} z^k
\end{aligned}$$

for $\mid z \mid < \rho$,

$$\begin{aligned}
h(z) \;=\; & \sum_{k=0}^{lm-1} \frac{A_{l,k}}{A_{0,k}} a_{k+(nm+c)l+cl} z^k + ON(n) \left(\frac{\mid z \mid^n}{(\rho-\epsilon)^{(lm+1)n}} + \right. \\
& \left. + \frac{1}{(\rho-\epsilon)^{n(l+1)m}} + \frac{1}{(\rho-\epsilon)^{(n+1)(l+1)m}} \right) \\
=\; & \sum_{k=0}^{lm-1} \frac{A_{l,k}}{A_{0,k}} a_{k+nlm+cl} z^k + ON(n) \left(\frac{\mid z \mid^n}{(\rho-\epsilon)^{(lm+1)n}} + \frac{1}{(\rho-\epsilon)^{n(l+1)m}} \right). \qquad (3.2.14)
\end{aligned}$$

As in (2.3.21) and (3.2.22), by choosing ϵ sufficiently small we can find η a positive number such that

$$\frac{|z|^n}{(\rho-\epsilon)^{(lm+1)n}} \le \left(\frac{1}{\rho^{lm}}-\eta\right)^n \tag{3.2.15}$$

and

$$\frac{1}{(\rho-\epsilon)^{n(l+1)m}} \le \left(\frac{1}{\rho^{lm}}-\eta\right)^n. \tag{3.2.16}$$

Thus from (3.2.14), (3.2.15) and (3.2.16) we have

$$h(z) = \sum_{k=0}^{lm-1} \frac{A_{l,k}}{A_{0,k}} a_{k+nlm+cl} z^k + ON(n)\left(\frac{1}{\rho^{lm}}-\eta\right)^n \tag{3.2.17}$$

where η is a positive number.

If we assume that in (ii) equality does not hold at more than $lm-1$ points say lm points then

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z_j, f) \mid^{1/n} < \frac{1}{\rho^{lm}} \qquad j=1,2,\cdots,lm$$

for $z_1, z_2, \cdots, z_{lm}$ with $|z_1|, |z_2|, \cdots, |z_{lm}| < \rho$. By the similar arguments as for the case $|z| > \rho$

$$\overline{\lim_{n\to\infty}} \mid h(z_j) \mid^{1/n} < \frac{1}{\rho^{lm}}. \qquad j=1,2,\ldots,lm$$

Now from (3.2.17)

$$\begin{aligned}\sum_{k=0}^{lm-1} \frac{A_{l,k}}{A_{0,k}} a_{k+nlm+cl} z_j^k &= ON(n)\left(\frac{1}{\rho^{lm}}-\eta\right)^n - h(z_j) \\ &= \beta_{j,n} \qquad (say)\end{aligned} \tag{3.2.18}$$

where, as for case (i)

$$\mid \beta_{j,n} \mid \le k_1 N(n)\left(\frac{1}{\rho^{lm}}-\eta_1\right)^n \tag{3.2.19}$$

for sufficiently large $n, k_1 > 1, \eta_1 > 0$ and $1 \le j \le lm$. Solving this system of equations (3.2.18) as earlier

$$\frac{A_{l,k}}{A_{0,k}} a_{k+nlm+cl} = \sum_{j=1}^{lm} c_j^{(k)} \beta_{j,n}$$

where c_j^k are appropiate constants independent of n. Hence from (3.2.19)

$$\overline{\lim_{n\to\infty}} \mid a_{k+nlm+cl} \mid^{1/k+nlm+cl}$$

$$\le \overline{\lim_{n\to\infty}} \left(\sum_{j=1}^{lm} c_j^{(k)} k_1 N(n)\left(\frac{1}{\rho^{lm}}-\eta_1\right)^n\right)^{1/k+nlm+cl}$$

$$= \overline{\lim_{n\to\infty}} \left(k_2 N(n) \left(\frac{1}{\rho^{lm}} - \eta_1 \right)^n \right)^{1/k+nlm+cl}$$

where $k_2 = \sum_{j=1}^{lm} c_j^{(k)} k_1 > 1, \qquad 0 \leq k \leq lm - 1$

thus

$$\overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} < \frac{1}{\rho}$$

which contradicts that $f \in A_\rho$. Hence

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z, f) \mid^{1/n} = \frac{1}{\rho^{lm}}$$

for all but at most $lm - 1$ distinct points in $0 < \mid z \mid < \rho$.

Remark 3.2.4 For $r = 1$ Theorem 3.2.2 reduces to corollary 3 of Theorem 7 [20].

Remark 3.2.5 For $r = 1, c = 0$ and $m = 1$ Theorem 3.2.2 reduces to Theorem 2.1.3.

From Theorem 3.2.1 and Theorem 3.2.2 we have

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z; f) \mid^{1/n} < \frac{\mid z \mid}{\rho^{1+lm}}$$

for at most lm distinct points in $\mid z \mid > \rho$. That is in $|z| > \rho^{1+lm}$

$$\overline{\lim_{n\to\infty}} \mid \Delta_{n-1,l,q}(z; f) \mid^{1/n} < B, \;\; B > 1$$

for at most lm distinct points. In other words we can say that

Remark 3.2.6 Let $f \in A_\rho, \rho > 1$ and $l \geq 1$ then the sequence $\{\Delta_{n-1,l,q}(z; f)\}_{n=1}^{\infty}$ can be bounded at most at lm distinct points in $|z| > \rho^{1+lm}$.

Corollary 3.2.2 *If f is analytic on $|z| \leq 1$ and if $\Delta_{n-1,l,q}(z; f)$ is uniformly bounded in every closed subdomain of $|z| < \rho^{1+lm}$ then f is analytic in $|z| < \rho$.*

Proof If f is analytic on $|z| \leq 1$. Let $f \in A_{\rho_1}$, then from Theorem 3.2.1, $g_{l,m} = K_{l,m}(R, \rho_1)$. Thus, by above Remark 3.2.6 $\{\Delta_{n-1,l,q}(z; f)\}_{n=1}^{\infty}$ can be bounded at most at lm distinct points in $|z| > {\rho_1}^{1+lm}$. Also it is given that $\Delta_{n-1,l,q}(z; f)$ is uniformly bounded in every closed subdomain of $|z| < \rho^{1+lm}$. Hence $\rho_1 < \rho$ is not possible. That is $\rho_1 \geq \rho$ which gives that f is analytic in $|z| < \rho$.

3.3 The object of this note is to consider roots of α^q in place of roots of unity, where $|\alpha| < \rho$. That is to study the polynomials $P_{n-1,r}(z,\alpha,f)$ which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{q-1} |Q_{n-1}^{(\nu)}(\omega^k) - f^{(\nu)}(\omega^k)|^2 \tag{3.3.1}$$

over all polynomials $Q_{n-1} \in \Pi_{n-1}$, where $\omega^q = \alpha^q$.

Lemma 3.3.1 *If $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A_\rho$, the unique polynomial $P_{n-1,r}(z,\alpha;f)$ which minimizes (3.3.1) over all polynomials $Q_{n-1} \in \Pi_{n-1}$, is given by*

$$P_{n-1,r}(z,\alpha;f) = \sum_{j=0}^{n-1} c_j(\alpha) z^j \tag{3.3.2}$$

where

$$c_j(\alpha) = \frac{1}{A_{0,j}(r)} \sum_{\lambda=0}^{\infty} A_{\lambda,j}(r) a_{j+\lambda q} \alpha^{\lambda q}, \qquad j = 0,1,\ldots,n-1 \tag{3.3.3}$$

and

$$A_{\lambda,j}(r) = \sum_{i=0}^{r-1} (j)_i (j+\lambda q)_i,$$

where $(j)_i = j(j-1),\ldots,(j-i+1)$ and $(j)_0 = 1$.

Before giving the proof of Lemma 3.3.1 we state and prove Lemma 3.3.2.

Let $f_0, f_1, \ldots, f_{r-1}$ be given functions in A_ρ and let $\{p_{\nu,j}\}_{j=0}^{n-1}$ $(\nu = 0,1,\ldots,r-1)$ be given real numbers. To each set of n numbers $\{p_{\nu,j}\}_{j=0}^{n-1}$ we define an operator $\mathcal{L}_\nu$ on the space of polynomials of degree $n-1$ such that if

$$Q_{n-1}(z) = \sum_{i=0}^{n-1} c_i z^i, \qquad \text{then} \qquad \mathcal{L}_\nu(Q_{n-1}(z)) = \sum_{i=0}^{n-1} c_i p_{\nu,i} z^i.$$

We now first find the polynomial $P_{n-1,r}(z,\alpha)$ which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{q-1} |\mathcal{L}_\nu Q_{n-1}(\alpha\omega^k) - f_\nu(\alpha\omega^k)|^2, \qquad \omega^q = 1 \tag{3.3.4}$$

over all polynomials $Q_{n-1} \in \Pi_{n-1}$. Let the polynomial interpolating $f_\nu(z)$ on the q roots of α^q be denoted by $L_{q-1}(z;\alpha;f_\nu)$. We set

$$L_{q-1}(z;\alpha;f_\nu) = \sum_{j=0}^{q-1} s_{\nu,j,\alpha}^{(q)} z^j, \qquad \nu = 0,1,\ldots,r-1 \tag{3.3.5}$$

where $s_{\nu,j,\alpha}^{(q)}$ depends upon f_ν and its value on q roots of α^q. We shall prove

Lemma 3.3.2 *The unique polynomial $P_{n-1,r}(z,\alpha;f)$ which minimizes (3.3.4) is given by*

$$P_{n-1,r}(z,\alpha;f)=\sum_{j=0}^{n-1}c_j(\alpha)z^j$$

where

$$c_j(\alpha)=\sum_{\nu=0}^{r-1}p_{\nu,j}s^{(q)}_{\nu,j,\alpha}\Big/\left\{\sum_{\nu=0}^{r-1}(p_{\nu,j})^2\right\},\qquad j=0,1,\ldots,n-1. \tag{3.3.6}$$

Proof: Observe that on using (3.3.5), we have

$$\begin{aligned}|\mathcal{L}_\nu Q_n(\alpha\omega^k)-f_\nu(\alpha\omega^k)|^2 &= |\mathcal{L}_\nu Q_{n-1}(\alpha\omega^k)-L_{q-1}(\alpha\omega^k;\alpha;f_\nu)|^2\\ &= |\sum_{j=0}^{n-1}c_jp_{\nu,j}z^j-\sum_{j=0}^{q-1}s^{(q)}_{\nu,j,\alpha}z^j|^2\\ &= |\sum_{j=0}^{q-1}d_{\nu,j}\alpha^j\omega^{kj}|^2\end{aligned}$$

where we have set

$$d_{\nu,j}=\begin{cases}s^{(q)}_{\nu,j,\alpha}-p_{\nu,j}c_j, & 0\le j\le n-1\\ s^{(q)}_{\nu,j,\alpha}, & n\le j\le q-1\end{cases} \tag{3.3.7}$$

By using the fact

$$\frac{1}{q}\sum_{k=0}^{q-1}\omega^{kp}=\begin{cases}1 & \text{if } p=sq, s\ge 0\\ 0 & \text{if } p\ne sq, s\ge 0\end{cases}$$

it follows that

$$\begin{aligned}\sum_{k=0}^{q-1}|\mathcal{L}_\nu Q_n(\alpha\omega^k)-f_\nu(\alpha\omega^k)|^2 &= \sum_{k=0}^{q-1}|\sum_{j=0}^{q-1}d_{\nu,j}\alpha^j\omega^{kj}|^2\\ &= \sum_{k=0}^{q-1}(\sum_{j=0}^{q-1}d_{\nu,j}\alpha^j\omega^{kj})(\sum_{i=0}^{q-1}\overline{d_{\nu,i}\alpha^i\omega^{ki}})\\ &= \sum_{k=0}^{q-1}\sum_{j=0}^{q-1}|d_{\nu,j}|^2|\alpha|^{2j}\\ &= q\sum_{j=0}^{q-1}|d_{\nu,j}|^2|\alpha|^{2j}.\end{aligned} \tag{3.3.8}$$

If we put

$$\left.\begin{aligned}&c_j=\rho_je^{i\theta_j}, \qquad j=0,1,\ldots,n-1\\ &s^{(q)}_{\nu,j,\alpha}=\sigma_{\nu,j,\alpha}e^{i\phi_{\nu,j,\alpha}}, \qquad j=0,1,\ldots,q-1;\ \nu=0,1,\ldots,r-1\end{aligned}\right\} \tag{3.3.9}$$

then from (3.3.7), it follows that

$$|d_{\nu,j}|^2=\begin{cases}p^2_{\nu,j}\rho^2_j+\sigma^2_{\nu,j,\alpha}-2p_{\nu,j}\rho_j\sigma_{\nu,j,\alpha}cos(\theta_j-\phi_{\nu,j,\alpha}), & j=0,1,\ldots,n-1\\ \sigma^2_{\nu,j,\alpha}, & j=n,\ldots,q-1.\end{cases}$$

Thus the problem (3.3.4) reduces to finding the minimum of the following

$$\sum_{\nu=0}^{r-1}\left\{\sum_{j=0}^{n-1}|\alpha|^{2j}p_{\nu,j}^2\rho_j^2+\sum_{j=0}^{q-1}|\alpha|^{2j}\sigma_{\nu,j,\alpha}^2-2\sum_{j=0}^{n-1}|\alpha|^{2j}p_{\nu,j}\rho_j\sigma_{\nu,j,\alpha}cos(\theta_j-\phi_{\nu,j,\alpha})\right\} \tag{3.3.10}$$

where ρ_j runs over the reals and $0\leq\theta_j\leq 2\pi$. Differentiating (3.3.10) with respect to ρ_j and θ_j we get the following system of equations to determine ρ_j and θ_j :

$$\left.\begin{array}{l}\rho_j\sum_{\nu=0}^{r-1}(p_{\nu,j})^2-\sum_{\nu=0}^{r-1}p_{\nu,j}\sigma_{\nu,j,\alpha}cos(\theta_j-\phi_{\nu,j,\alpha})=0\\ \sum_{\nu=0}^{r-1}p_{\nu,j}\sigma_{\nu,j,\alpha}sin(\theta_j-\phi_{\nu,j,\alpha})=0,\end{array}\right\} \tag{3.3.11}$$

Solving these equations we have

$$\rho_j\sum_{\nu=0}^{r-1}(p_{\nu,j})^2-\sum_{\nu=0}^{r-1}p_{\nu,j}\sigma_{\nu,j,\alpha}e^{-i(\theta_j-\phi_{\nu,j,\alpha})}=0$$

or,

$$\rho_je^{i\theta_j}\sum_{\nu=0}^{r-1}(p_{\nu,j})^2-\sum_{\nu=0}^{r-1}p_{\nu,j}\sigma_{\nu,j,\alpha}e^{i\phi_{\nu,j,\alpha}}=0$$

i.e.

$$c_j\sum_{\nu=0}^{r-1}(p_{\nu,j})^2-\sum_{\nu=0}^{r-1}p_{\nu,j}s_{\nu,j,\alpha}^{(q)}=0$$

which gives the result.

Proof of Lemma 3.3.1 : From (3.3.4) it follows that (3.3.1) reduces to

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{q-1}|\sum_{j=0}^{n-1}(j)_\nu c_j(\alpha^j\omega^{jk})-f_\nu(\alpha\omega^k)|^2,\qquad \omega^q=1$$

where $f_\nu(z)=z^\nu f^{(\nu)}(z),\nu=0,1,\ldots,r-1$. The result now follows from Lemma 3.3.2 on taking

$$p_{\nu,j}=(j)_\nu,\qquad \nu=0,1,\ldots,r-1,\ j=0,1,\ldots,n-1.$$

Since $f\in A_\rho$, we have

$$f_\nu(z)=z^\nu f^{(\nu)}(z)=\frac{\nu!}{2\pi i}\int_\Gamma\frac{f(t)z^\nu}{(t-z)^{\nu+1}}dt$$

where Γ is the circle $|t|=R<\rho$ such that $|\alpha|<R$. Then

$$f_\nu(\alpha\omega^k)=\frac{\nu!}{2\pi i}\int_\Gamma\frac{f(t)(\alpha\omega^k)^\nu}{(t-(\alpha\omega^k))^{\nu+1}}dt,\qquad \omega^q=1.$$

Now if

$$g(\alpha\omega^k)=\frac{\nu!(\alpha\omega^k)^\nu}{(t-(\alpha\omega^k))^{\nu+1}}dt,\qquad \omega^q=1.$$

then by Hermite interpolating formula we have

$$L_{q-1}(z,\alpha,g)=\frac{1}{2\pi i}\int_{\Gamma'}\frac{\nu! y^\nu}{(t-y)^{\nu+1}}\frac{y^q-z^q}{(y-z)(y^q-\alpha^q)}dy$$

where $\Gamma' : |y| = R' < R$ where R' is such that $|\alpha| < R'$.

$$\begin{aligned}
L_{q-1}(z,\alpha,g) &= \frac{1}{2\pi i}\int_{\Gamma'}\frac{\nu! y^\nu}{(t-y)^{\nu+1}}\frac{y^q-z^q}{(y-z)(y^q-\alpha^q)}dy \\
&= -Residue\left(\frac{\nu! y^\nu}{(t-y)^{\nu+1}}\frac{y^q-z^q}{(y-z)(y^q-\alpha^q)}, y=t\right) \\
&= -\frac{1}{\nu!}\lim_{y\to t}\left(\frac{d^\nu}{dy^\nu}(y-t)^{\nu+1}\frac{\nu! y^\nu}{(t-y)^{\nu+1}}\frac{y^q-z^q}{(y-z)(y^q-\alpha^q)}\right) \\
&= (-1)^{\nu+2}\frac{d^\nu}{dt^\nu}\left(\frac{t^\nu(t^q-z^q)}{(t-z)(t^q-\alpha^q)}\right) \\
&= (-1)^\nu\sum_{k=0}^{q-1}\left(\frac{d^\nu}{dt^\nu}\frac{t^{\nu+q-k-1}z^k}{(t^q-\alpha^q)}\right).
\end{aligned}$$

Thus,

$$L_{q-1}(z;\alpha;f_\nu)=\frac{1}{2\pi i}\int_{\Gamma'}f(t)\left\{(-1)^\nu\sum_{j=0}^{q-1}\frac{d^\nu}{dt^\nu}\left(\frac{t^{q+\nu-j-1}}{t^q-\alpha^q}\right)z^j\right\}dt.$$

Also

$$\begin{aligned}
(-1)^\nu\frac{d^\nu}{dt^\nu}\left(\frac{t^{q+\nu-j-1}}{t^q-\alpha^q}\right) &= (-1)^\nu\frac{d^\nu}{dt^\nu}\left(\sum_{\lambda=0}^{\infty}\frac{\alpha^{\lambda q}t^{\nu-j-1}}{t^{\lambda q}}\right) \\
&= (-1)^\nu\sum_{\lambda=0}^{\infty}\alpha^{\lambda q}\frac{d^\nu}{dt^\nu}t^{-\lambda q+\nu-j-1} \\
&= (-1)^\nu\sum_{\lambda=0}^{\infty}\alpha^{\lambda q}(-\lambda q+\nu-j-1) \\
&\quad (-\lambda q+\nu-j-1-1)\ldots \\
&\quad (-\lambda q+\nu-j-1-\nu+1)t^{-\lambda q-j-1} \\
&= \sum_{\lambda=0}^{\infty}\alpha^{\lambda q}(\lambda q-\nu+j+1)(\lambda q-\nu+j+2)\ldots \\
&\quad (\lambda q+j)t^{-\lambda q-j-1} \\
&= \sum_{\lambda=0}^{\infty}\alpha^{\lambda q}(j+\lambda q)_\nu t^{-\lambda q-j-1} \\
&= t^{q-1-j}\sum_{\lambda=0}^{\infty}\frac{(j+\lambda q)_\nu\alpha^{q\lambda}}{t^{(\lambda+1)q}}.
\end{aligned}$$

Thus

$$L_{q-1}(z;\alpha;f_\nu)=\frac{1}{2\pi i}\int_{\Gamma'}f(t)\left\{\sum_{j=0}^{q-1}\hat{S}_{\nu,j}(t)t^{q-1-j}z^j\right\}dt,$$

where

$$\hat{S}_{\nu,j}(t)=\sum_{\lambda=0}^{\infty}\frac{(j+\lambda q)_\nu\alpha^{q\lambda}}{t^{(\lambda+1)q}}.$$

Hence

$$L_{q-1}(z;\alpha;f_\nu)=\sum_{j=0}^{q-1}\sum_{\lambda=0}^{\infty}(j+\lambda q)_\nu\alpha^{\lambda q}a_{j+\lambda q}z^j$$

and

$$s^{(q)}_{\nu,j,\alpha}=\sum_{\lambda=0}^{\infty}(j+\lambda q)_\nu\alpha^{\lambda q}a_{j+\lambda q}.$$

Whence

$$\begin{aligned} c_j(\alpha) &= \frac{\sum_{\nu=0}^{r-1}(j)_\nu\sum_{\lambda=0}^{\infty}(j+\lambda q)_\nu\alpha^{\lambda q}a_{j+\lambda q}}{\sum_{\nu=0}^{r-1}(j)_\nu(j)_\nu} \\ &= \frac{1}{A_{0,j}(r)}\sum_{\lambda=0}^{\infty}A_{\lambda,j}(r)a_{j+\lambda q}\alpha^{\lambda q}, \qquad j=0,1,\ldots,n-1, \end{aligned}$$

where

$$A_{\lambda,j}(r)=\sum_{i=0}^{r-1}(j)_i(j+\lambda q)_i, \qquad (j)_i=j(j-1),\ldots,(j-i+1).$$

giving the required result.

Now let $\alpha,\beta\in D_\rho$ be two arbitrary points, and let $f\in A_\rho$. Further, we assume

$$q=nm+c, \qquad m\geq 1, \qquad 0\leq c<m,$$

$$s=lq+p, \qquad l\geq 1, r_1\leq p/n<1; \qquad p/n=r_1+\mathcal{O}(\frac{1}{n}),$$

and

$$t=bs+d, \qquad b\geq 1, r_2\leq d/n<1; \qquad d/n=r_2+\mathcal{O}(\frac{1}{n}),$$

where p and d are some integers, and $r_1,r_2\in[0,1)$ are given constants.

Let $P_{n-1,r}(z,\alpha,f)$ is the polynomial which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{q-1}|Q^{(\nu)}_{n-1}(\omega^k)-f^{(\nu)}(\omega^k)|^2, \qquad \omega^q=\alpha^q$$

over all polynomials $Q_{n-1}\in\Pi_{n-1}$.

Similarly let $P_{s-1,r}(z,\beta,f)$ is the polynomial which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{t-1}|Q^{(\nu)}_{s-1}(\omega^k)-f^{(\nu)}(\omega^k)|^2, \qquad \omega^t=\beta^t$$

over all polynomials $Q_{s-1}\in\Pi_{s-1}$.

Let us denote

$$\Delta^{\alpha,\beta}_{n-1,s,r}(z;f)=P_{n-1,r}(z,\alpha;f)-P_{n-1,r}(z,\alpha;P_{s-1,1}(z,\beta;f)),$$

$$g_{\alpha,\beta}(R) = \overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta^{\alpha,\beta}_{n-1,s,r}(z; f)|^{1/n}$$

and

$$K_{\alpha,\beta}(R,\rho) = \begin{cases} max\left(|\frac{\alpha}{\rho}|^{m(l+1)}, |\frac{\alpha}{\rho}|^{ml}|\frac{R}{\rho}|^{r_1}, |\frac{\beta}{\rho}|^{(lm+r_1)b+r_2}\right) & \text{if } 0 < |z| < \rho \\ max\left(|\frac{\alpha}{\rho}|^{ml}|\frac{R}{\rho}|, |\frac{\beta}{\rho}|^{(lm+r_1)b+r_2}|\frac{R}{\rho}|\right) & \text{if } |z| \geq \rho \end{cases}$$

then

Theorem 3.3.1 *If $t = t_n = sb + d, d = d_n = r_2 n + \mathcal{O}(1), 0 \leq r_2 < 1, s = s_n = lq + p, p = p_n = r_1 n + \mathcal{O}(1), 0 \leq r_1 < 1, q = q_n = mn + c, 0 \leq c < m$ and for each $\alpha, \beta \in D_\rho$ if $|\alpha/\rho|^{lm} \neq |\beta/\rho|^{(lm+r_1)b+r_2}$ and for $r_1 \neq 0$ if $|\alpha/\rho|^{m(l+1)} \neq |\beta/\rho|^{(lm+r_1)b+r_2}$ then for each $f \in A_\rho$*

$$g_{\alpha,\beta}(R) = K_{\alpha,\beta}(R,\rho), \qquad R > 0.$$

Note that for $r = 1, q = n, t = s$,

$$P_{n-1,r}(z,\alpha; f) = L_{n-1}(z,\alpha; f)$$

and

$$P_{s-1,1}(z,\beta; f) = L_{s-1}(z,\beta; f).$$

Remark 3.3.1 For the special case $r = 1, q = n, t = s$ Theorem 3.3.1 reduces to Theorem 3.1.4.

Next, for $\alpha = 1, \beta = 0, p = 0, t = s$

$$P_{n-1,r}(z,\alpha; f) = P_{n-1,r}(z; f)$$

and

$$P_{s-1,1}(z,\beta; f) = S_{s-1}(z; f) = S_{lq-1}(z, f)$$

Thus

$$P_{n-1,r}(z, P_{s-1,1}(z,\beta; f)) = P_{n-1,r}(S_{lq-1}(z, f))$$

and from (3.1.1) we have

$$P_{n-1,r}(z; S_{lq-1}(z; f)) = \sum_{k=0}^{n-1}\sum_{j=0}^{l-1} \frac{A_{j,k}(r)}{A_{0,k}(r)} a_{k+qj} z^k.$$

Thus,

Remark 3.3.2 For the special case $\alpha = 1, \beta = 0, p = 0, t = s$ Theorem 3.3.1 reduces to Theorem 3.2.1.

Proof : Here and after we consider $A_{j,k} = A_{j,k}(r)$. Since for $r = 1$ $A_{j,k} = 1$. from (3.3.2) and (3.3.3)

$$\begin{aligned} P_{s-1,1}(z,\beta;f) &= \sum_{j=0}^{\infty}\sum_{k=0}^{s-1} a_{k+tj}\beta^{tj}z^k \\ &= \sum_{k=0}^{s-1} d_k z^k, \qquad \text{where} \qquad d_k = \sum_{j=0}^{\infty} a_{k+tj}\beta^{tj} \\ &= \sum_{k=0}^{lq+p-1} d_k z^k \\ &= \sum_{j=0}^{l-1}\sum_{k=0}^{q-1} d_{k+qj}z^{k+qj} + \sum_{k=0}^{p-1} d_{k+ql}z^{k+ql} \end{aligned}$$

hence

$$\begin{aligned} P_{n-1,r}(z,\alpha;P_{s-1,1}(z,\beta;f)) &= P_{n-1,r}\left(z,\alpha;\left(\sum_{j=0}^{l-1}\sum_{k=0}^{q-1} d_{k+qj}z^{k+qj} + + \sum_{k=0}^{p-1} d_{k+ql}z^{k+ql}\right)\right) \\ &= \sum_{j=0}^{l-1}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}} d_{k+qj}\alpha^{qj}z^k + \sum_{k=0}^{p-1}\frac{A_{l,k}}{A_{0,k}} d_{k+ql}\alpha^{ql}z^k \\ &= \sum_{j=0}^{l-1}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}}\sum_{i=0}^{\infty} a_{k+ti+qj}\beta^{ti}\alpha^{qj}z^k + \\ &\quad + \sum_{k=0}^{p-1}\frac{A_{l,k}}{A_{0,k}}\sum_{i=0}^{\infty} a_{k+ti+ql}\beta^{ti}\alpha^{ql}z^k \end{aligned} \tag{3.3.20}$$

also from (3.3.2)

$$P_{n-1,r}(z,\alpha;f) = \sum_{j=0}^{\infty}\sum_{k=0}^{n-1}\frac{A_{j,k}}{A_{0,k}} a_{k+qj}\alpha^{qj}z^k$$

this together with (3.3.20) gives

$$\Delta^{\alpha,\beta}_{n-1,s,r}(z;f) = \sum_{k=0}^{n-1} D_{k,n}z^k \tag{3.3.21}$$

where

$$D_{k,n} = \begin{cases} -\sum_{i=0}^{\infty}\frac{A_{l,k}}{A_{0,k}} a_{k+ti+ql}\beta^{ti}\alpha^{ql} - \sum_{i=0}^{\infty}\sum_{j=0}^{l-1}\frac{A_{j,k}}{A_{0,k}} a_{k+ti+qj}\beta^{ti}\alpha^{qj} & \\ \quad + \sum_{j=0}^{\infty}\frac{A_{j,k}}{A_{0,k}} a_{k+qj}\alpha^{qj} & \text{for } 0 \le k \le p_n - 1 \\ -\sum_{i=0}^{\infty}\sum_{j=0}^{l-1}\frac{A_{j,k}}{A_{0,k}} a_{k+ti+qj}\beta^{ti}\alpha^{qj} + \sum_{j=0}^{\infty}\frac{A_{j,k}}{A_{0,k}} a_{k+qj}\alpha^{qj} & \\ & \text{for } p_n \le k \le n-1 \end{cases}$$

$$= \begin{cases} \sum_{j=0}^{\infty}\frac{A_{j,k}}{A_{0,k}} a_{k+qj}\alpha^{qj} - \sum_{i=0}^{\infty}\sum_{j=0}^{l}\frac{A_{j,k}}{A_{0,k}} a_{k+ti+qj}\beta^{ti}\alpha^{qj} & \\ & \text{for } 0 \le k \le p_n - 1 \\ \sum_{j=0}^{\infty}\frac{A_{j,k}}{A_{0,k}} a_{k+qj}\alpha^{qj} - \sum_{i=0}^{\infty}\sum_{j=0}^{l-1}\frac{A_{j,k}}{A_{0,k}} a_{k+ti+qj}\beta^{ti}\alpha^{qj} & \\ & \text{for } p_n \le k \le n-1 \end{cases}$$

for $0 \le k \le p_n - 1$ let $\epsilon > 0$ be too small that

$$(\rho/(\rho-\epsilon))^{r_1} max\left\{|\frac{\alpha}{\rho-\epsilon}|^{m(l+2)}, |\frac{\beta}{\rho-\epsilon}|^{(lm+r_1)b+r_2}|\frac{\alpha}{\rho-\epsilon}|^{m}, |\frac{\beta}{\rho-\epsilon}|^{2((lm+r_1)b+r_2)}\right\} <$$

$$max\left\{|\frac{\alpha}{\rho}|^{m(l+1)}, |\frac{\beta}{\rho}|^{(lm+r_1)b+r_2}\right\} = \Lambda_1.$$

Thus,

$$\begin{aligned}
D_{k,n} &= \sum_{j=0}^{\infty}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}\alpha^{qj} - \sum_{j=0}^{l}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}\alpha^{qj} - \\
&\quad -\sum_{j=0}^{l}\frac{A_{j,k}}{A_{0,k}}a_{k+t+qj}\beta^{t}\alpha^{qj} - \sum_{i=2}^{\infty}\sum_{j=0}^{l}\frac{A_{j,k}}{A_{0,k}}a_{k+ti+qj}\beta^{ti}\alpha^{qj} \\
&= \sum_{j=l+1}^{\infty}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}\alpha^{qj} - \frac{A_{0,k}}{A_{0,k}}a_{k+t}\beta^{t} - \\
&\quad \sum_{j=1}^{l}\frac{A_{j,k}}{A_{0,k}}a_{k+t+qj}\beta^{t}\alpha^{qj} - \sum_{i=2}^{\infty}\sum_{j=0}^{l}\frac{A_{j,k}}{A_{0,k}}a_{k+ti+qj}\beta^{ti}\alpha^{qj} \\
&= \frac{A_{l+1,k}}{A_{0,k}}a_{k+q(l+1)}\alpha^{q(l+1)} - a_{k+t}\beta^{t} + \\
&\quad +\mathcal{O}N(n)\left(\frac{|\alpha|^{q(l+2)}}{(\rho-\epsilon)^{q(l+2)+k}} + \frac{|\beta|^{t}|\alpha|^{q}}{(\rho-\epsilon)^{k+t+q}} + \frac{|\beta|^{2t}}{(\rho-\epsilon)^{k+2t}}\right) \\
&= \frac{A_{l+1,k}}{A_{0,k}}a_{k+q(l+1)}\alpha^{q(l+1)} - a_{k+t}\beta^{t} + \rho^{-k}\mathcal{O}(N(n)(\sigma\Lambda_1)^{n}) \qquad (3.3.22)
\end{aligned}$$

where $0 < \sigma < 1$ and $N(n)$ is quantity dependent of n such that $\lim_{n\to\infty}(N(n))^{1/n} = 1$, further $N(n)$ may not be same at each occurence.

Similarly for $p_n \le k \le n-1$ let $\epsilon > 0$ be so small that

$$(\rho/(\rho-\epsilon)) max\left\{|\frac{\alpha}{\rho-\epsilon}|^{m(l+1)}, |\frac{\beta}{\rho-\epsilon}|^{(lm+r_1)b+r_2}|\frac{\alpha}{\rho-\epsilon}|^{m}, |\frac{\beta}{\rho-\epsilon}|^{2((lm+r_1)b+r_2)}\right\} <$$

$$max\left\{|\frac{\alpha}{\rho}|^{ml}, |\frac{\beta}{\rho}|^{(lm+r_1)b+r_2}\right\} = \Lambda_2.$$

Thus,

$$\begin{aligned}
D_{k,n} &= \sum_{j=0}^{\infty}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}\alpha^{qj} - \sum_{i=0}^{\infty}\sum_{j=0}^{l-1}\frac{A_{j,k}}{A_{0,k}}a_{k+ti+qj}\beta^{ti}\alpha^{qj} \\
&= \sum_{j=0}^{\infty}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}\alpha^{qj} - \sum_{j=0}^{l-1}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}\alpha^{qj} - \\
&\quad -\sum_{j=0}^{l-1}\frac{A_{j,k}}{A_{0,k}}a_{k+t+qj}\beta^{t}\alpha^{qj} - \sum_{i=2}^{\infty}\sum_{j=0}^{l-1}\frac{A_{j,k}}{A_{0,k}}a_{k+ti+qj}\beta^{ti}\alpha^{qj} \\
&= \sum_{j=l}^{\infty}\frac{A_{j,k}}{A_{0,k}}a_{k+qj}\alpha^{qj} - \frac{A_{0,k}}{A_{0,k}}a_{k+t}\beta^{t} -
\end{aligned}$$

$$\sum_{j=1}^{l-1} \frac{A_{j,k}}{A_{0,k}} a_{k+t+qj} \beta^t \alpha^{qj} - \sum_{i=2}^{\infty} \sum_{j=0}^{l-1} \frac{A_{j,k}}{A_{0,k}} a_{k+ti+qj} \beta^{ti} \alpha^{qj}$$

$$\begin{aligned}
&= \frac{A_{l,k}}{A_{0,k}} a_{k+ql} \alpha^{ql} - a_{k+t} \beta^t + \\
&\quad + \mathcal{O} N(n) \left(\frac{|\alpha|^{q(l+1)}}{(\rho-\epsilon)^{q(l+1)+k}} + \frac{|\beta|^t |\alpha|^q}{(\rho-\epsilon)^{k+t+q}} + \frac{|\beta|^{2t}}{(\rho-\epsilon)^{k+2t}} \right) \\
&= \frac{A_{l,k}}{A_{0,k}} a_{k+ql} \alpha^{ql} - a_{k+t} \beta^t + \rho^{-k} \mathcal{O}(N(n)(\sigma \Lambda_2)^n)
\end{aligned} \tag{3.3.23}$$

hence

$$\begin{aligned}
\Delta^{\alpha,\beta}_{n-1,s,r}(z;f) &= \alpha^{q_n(l+1)} \sum_{k=0}^{p_n-1} \frac{A_{l+1,k}}{A_{0,k}} a_{k+q_n(l+1)} z^k + \alpha^{q_n l} \sum_{k=p_n}^{n-1} \frac{A_{l,k}}{A_{0,k}} a_{k+q_n l} z^k \\
&\quad - \beta^{t_n} \sum_{k=0}^{n-1} a_{k+t_n} z^k + R_n(z)
\end{aligned}$$

where

$$\begin{aligned}
R_n(z) &= \mathcal{O}\left(N(n) \sigma^n \Lambda_1^n \sum_{k=0}^{p_n-1} |z/\rho|^k + N(n) \sigma^n \Lambda_2^n \sum_{k=p_n}^{n-1} |z/\rho|^k \right) \\
&= \begin{cases} \mathcal{O}\left(N(n)(\sigma max(\Lambda_1 + \Lambda_2 |z/\rho|^{r_1}))^n\right) & \text{if } |z| < \rho \\ \mathcal{O}\left(N(n)(\sigma max(\Lambda_1 |z/\rho|^{r_1} + \Lambda_2 |z/\rho|))^n\right) & \text{if } |z| \geq \rho \end{cases}
\end{aligned}$$

hence

$$\Delta^{\alpha,\beta}_{n-1,s,r}(z;f) = \begin{cases} \mathcal{O} N(n) \left(|\frac{\alpha}{(\rho-\epsilon)}|^{q(l+1)} + |\frac{\alpha}{(\rho-\epsilon)}|^{ql} |\frac{z}{(\rho-\epsilon)}|^p + |\frac{\beta}{(\rho-\epsilon)}|^t \right) + R_n(z) \\ \qquad\qquad \text{if } 0 < |z| < \rho \\ \mathcal{O} N(n) \left(|\frac{\alpha}{(\rho-\epsilon)}|^{q(l+1)} |\frac{z}{(\rho-\epsilon)}|^p + |\frac{\alpha}{(\rho-\epsilon)}|^{ql} |\frac{z}{(\rho-\epsilon)}|^n + |\frac{\beta}{(\rho-\epsilon)}|^t |\frac{z}{(\rho-\epsilon)}|^n \right) \\ + R_n(z) \qquad\qquad \text{if } |z| \geq \rho \end{cases}$$

on taking n^{th} root which yields

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta^{\alpha,\beta}_{n-1,s,r}(z;f)|^{1/n} \leq K_{\alpha,\beta}(R, \rho - \epsilon)$$

since ϵ is arbitrarily small hence

$$g_{\alpha,\beta}(R) \leq K_{\alpha,\beta}(R,\rho).$$

For the opposite inequality to show that $g_{\alpha,\beta}(R) \geq K_{\alpha,\beta}(R,\rho)$.

Now from (3.3.5) with Caushi's formula we have

$$D_{k,n} = \frac{1}{2\pi i} \int_{|z|=R} \frac{\Delta^{\alpha,\beta}_{n-1,s,r}(z;f)}{z^{k+1}} dz$$

and therefore

$$R^k|D_{k,n}| \le \max_{|z|=R} |\Delta^{\alpha,\beta}_{n-1,s,r}(z;f)|, \qquad 0 \le k \le n-1, \qquad R > 0. \tag{3.3.24}$$

Now $k+q_nl = k+mln+lc$ and $k+t = k+bs+d = k+(lq+p)b+d = k+(l(mn+c)+p)b+d = k + lmbn + lcb + pb + d$. It is clear that there exists an integer $C > 0$ such that for $n - C \le k \le n-1$, the sequences $\{k + q_nl\}$ and $\{k + t_n\}$ takes all positive integer values Since $p_n < n - C$ for sufficiently large n and $|\frac{\alpha}{\rho}|^{ml} \ne |\frac{\beta}{\rho}|^{(lm+r_1)b+r_2}$, hence from (3.3.23)

$$\overline{\lim_{n\to\infty}}\{\max_{n-C\le k\le n-1} |D_{k,n}|\}^{1/n} = \frac{1}{(\rho-\epsilon)} max(|\frac{\alpha}{(\rho-\epsilon)}|^{ml}, |\frac{\beta}{(\rho-\epsilon)}|^{(lm+r_1)b+r_2})$$

with (3.3.24) which gives

$$\frac{R}{(\rho-\epsilon)} max(|\frac{\alpha}{(\rho-\epsilon)}|^{ml}, |\frac{\beta}{(\rho-\epsilon)}|^{(lm+r_1)b+r_2}) \le g_{\alpha,\beta}(R). \tag{3.3.25}$$

Similarly we can choose $C > 0$ such that the sequences $\{k + q_nl\}$ and $\{k + t_n\}$ assumes all positive integer values for $p_n \le k \le p_n + C$ and $p_n + C < n$ for sufficiently large n, hence from (3.3.23),

$$\overline{\lim_{n\to\infty}}\{\max_{p_n\le k\le p_n+C} |D_{k,n}|\}^{1/n} = \frac{1}{(\rho-\epsilon)^{r_1}} max(|\frac{\alpha}{(\rho-\epsilon)}|^{ml}, |\frac{\beta}{(\rho-\epsilon)}|^{(lm+r_1)b+r_2})$$

hence from (3.3.24)

$$|\frac{R}{(\rho-\epsilon)}|^{r_1} max(|\frac{\alpha}{(\rho-\epsilon)}|^{ml}, |\frac{\beta}{(\rho-\epsilon)}|^{(lm+r_1)b+r_2}) \le g_{\alpha,\beta}(R). \tag{3.3.26}$$

For the case $r_1 = 0$ from (3.3.25) and (3.3.26) we have

$$g_{\alpha,\beta}(R) \ge K_{\alpha,\beta}(R, (\rho-\epsilon)).$$

Let now $r_1 > 0$. As $k+t_n = k+lmn+lc+p+d$ and $k+q_n(l+1) = k+m(l+1)n+(l+1)c$. choose $C > 0$ such that $\{k+t_n\}$ and $\{k+q_n(l+1)\}$ for $0 \le k \le C$ assume all positive integer values. But for n sufficiently large, we have $C < p_n$, and since $|\frac{\alpha}{\rho}|^{m(l+1)} \ne |\frac{\beta}{\rho}|^{(lm+r_1)b+r_2}$, thus from (3.3.22) we have

$$\overline{\lim_{n\to\infty}}\{\max_{0\le k\le C} |D_{k,n}|\}^{1/n} = max(|\frac{\alpha}{(\rho-\epsilon)}|^{m(l+1)}, |\frac{\beta}{(\rho-\epsilon)}|^{(lm+r_1)b+r_2}).$$

This together with (3.3.24) gives

$$max(|\frac{\alpha}{(\rho-\epsilon)}|^{m(l+1)}, |\frac{\beta}{(\rho-\epsilon)}|^{(lm+r_1)b+r_2}) \le g_{\alpha,\beta}(R). \tag{3.3.27}$$

Similarly if $C > 0$ is such that the sequence $\{k+t_n\}$ and $\{k+q_n(l+1)\}$ for $p_n - C \le k \le p_n - 1$ assumes all positive integer values, from (3.3.22) we obtain

$$\overline{\lim_{n\to\infty}}\{\max_{p_n-C\le k\le p_n-1}|D_{k,n}|\}^{1/n} = \frac{1}{(\rho-\epsilon)^{r_1}}max(|\frac{\alpha}{(\rho-\epsilon)}|^{m(l+1)}, |\frac{\beta}{(\rho-\epsilon)}|^{(lm+r_1)b+r_2}).$$

This together with (3.3.24) gives

$$|\frac{R}{(\rho-\epsilon)}|^{r_1} max(|\frac{\alpha}{(\rho-\epsilon)}|^{m(l+1)}, |\frac{\beta}{(\rho-\epsilon)}|^{(lm+r_1)b+r_2}) \le g_{\alpha,\beta}(R). \tag{3.3.28}$$

From (3.3.25),(3.3.26), (3.3.27) and (3.3.28) it follows that for $0 < r_1 < 1, 0 \le r_2 < 1$, for $0 < R < \rho$ we have

$$max\left\{|\frac{\alpha}{(\rho-\epsilon)}|^{m(l+1)}, |\frac{R}{(\rho-\epsilon)}|^{r_1}|\frac{\alpha}{(\rho-\epsilon)}|^{ml}, |\frac{\beta}{(\rho-\epsilon)}|^{(ml+r_1)b+r_2}\right\} \le g_{\alpha,\beta}(R)$$

and for $R \ge \rho$ we have

$$|\frac{R}{(\rho-\epsilon)}| max\left\{|\frac{\alpha}{(\rho-\epsilon)}|^{ml}, |\frac{\beta}{(\rho-\epsilon)}|^{(ml+r_1)b+r_2}\right\} \le g_{\alpha,\beta}(R) \qquad \text{for} \qquad R \ge \rho.$$

Since ϵ is arbitrary small we have

$$K_{\alpha,\beta}(R,\rho) \le g_{\alpha,\beta}(R)$$

which completes the proof.

Chapter 4

WALSH OVERCONVERGENCE USING AVERAGES OF LEAST SQUARE APPROXIMATING POLYNOMIALS

4.1 Let $A_\rho(1 < \rho < \infty)$ be the class of functions $f(z)$, analytic in $|z| < \rho$ and having a singularity on the circle $|z| = \rho$. L. Yuanren [25] and M.P.Stojanova [50] generalised Theorem 1.1.1, an extension of Walsh's theorem, by considering $D_\rho = \{z \in C; |z| < \rho\}, \Gamma_\rho = \{z \in C; |z| = \rho\}$. That is A_ρ denote the set of all functions $f(z)$ which are analytic in D_ρ but not on Γ_ρ. Let $\alpha, \beta \in D_\rho$ and for any positive integer s and $n(s > n)$ let $L_{n-1}(z, \alpha; f)$ and $L_{s-1}(z, \beta; f)$ denote the Lagrange interpolants of f in the zeros of $z^n - \alpha^n$ and $z^s - \beta^s$ respectively. With above notations L. Yuanren [25] proved

Theorem 4.1.1 [25] *If $s = s_n = ln + p, p = p_n = r_1 n + O(1), 0 \leq r_1 < 1, p \geq 0$ then for each $f \in A_\rho$ and for each $\alpha, \beta \in D_\rho$, we have*

$$\overline{\lim_{n\to\infty}} |\Delta_{n,s}^{\alpha,\beta}(z; f)|^{1/n} = 0, \qquad \forall\ |z| < \tau,$$

where

$$\Delta_{n,s}^{\alpha,\beta}(z; f) = L_{n-1}(z, \alpha; f) - L_{n-1}(z, \alpha, L_{s-1}(z, \beta; f))$$

and

$$\tau = \rho / max\{|\alpha/\rho|^l, |\beta/\rho|^{l+r_1}\}.$$

More precisely for any R with $\rho < R < \infty$, we have

$$\overline{\lim_{n\to\infty}} \{\max_{z\in D_R} |\Delta_{n,s}^{\alpha,\beta}(z; f)|^{1/n}\} \leq R/\tau.$$

When $\alpha = 1, \beta = 0$ and $s = ln$, the above result yields a result of Cavaretta et al [12], which itself is a generalisation of a theorem of Walsh [58,p.153].

M.P.Stojanova [50] obtained more precise theorem for the difference $\Delta_{n,s}^{\alpha,\beta}$:

Theorem 4.1.2 [50] *With the hypothesis of Theorem 4.1.1, if* $|\alpha/\rho|^l \neq |\beta/\rho|^{l+r_1}$ *and for* $r_1 \neq 0$ *if* $|\alpha/\rho|^{l+1} \neq |\beta/\rho|^{l+r_1}$, *then*

$$\overline{\lim_{n\to\infty}}\{\max_{|z|=R}|\Delta_{n,s}^{\alpha,\beta}(z;f)|^{1/n}\} = K_\rho(R), \qquad R > 0,$$

where

$$K_\rho(R) = \begin{cases} (R/\rho)max\{|\alpha/\rho|^l, |\beta/\rho|^{l+r_1}\} & for \ \ R \geq \rho \\ max\{|\alpha/\rho|^{l+1}, |\alpha/\rho|^l(R/\rho)^{r_1}, |\beta/\rho|^{l+r_1}\} & for \ \ 0 < R < \rho \end{cases}$$

As a particular case $\alpha = 1, \beta = 0$ and $s_n = ln$, Theorem 4.1.2 reduces to Theorem 2.1.2.

In this chapter we extend a few results of Chapter 2 by considering average of least square approximating polynomials. Motivated by Cavaretta et al [10] in section 4.2 and 4.3 we give two Lemmas and two theorems extending the result of Theorem 2.2.1 and Theorem 2.3.2. In section 4.4 and 4.5 motivated by the work of M.P.Stojanova [50] roots of α^n are considered , where $|\alpha| < \rho$ and Theorem 4.3.1 is generalised.

4.2 For positive integers m and n set

$$\omega_{s,k} = e^{\frac{2\pi i}{mn}(km+s)}, \tag{4.2.1}$$

for $k = 0, \ldots, n-1$ and $s = 0, \ldots, m-1$. It is clear that $\omega_{s,k}$ is an mn^{th} root of unity. Also, every such root is given by $\omega_{s,k}$ for a unique pair (s,k) satisfying $0 \leq s \leq m; 0 \leq k \leq n-1$. In addition, $\Omega_s := (\omega_{s,k})^n$ is a m^{th} root of unity for $k = 0, \ldots, n-1$. Let $f \in A_\rho$ and

$$G_{n-1,r}(z;f) = \frac{1}{m}\sum_{s=0}^{m-1} G_{n-1,r}^s(z;f) \tag{4.2.2}$$

where for each $s = 0, \ldots, m-1$, $G_{n-1,r}^s(z;f)$ is polynomial of degree $n-1$ which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{n-1}|Q_{n-1}^{(\nu)}(\omega_{s,k}) - f^{(\nu)}(\omega_{s,k})|^2 \tag{4.2.3}$$

where r is a fixed integer and $\omega_{s,k}$ are given by (4.2.1), over all polynomials Q_{n-1} of degree $\leq n-1$.

In this section we determine explicit expression for the polynomial $G_{n-1,r}(z; f)$ for which first we find expression for $G^s_{n-1,r}(z; f)$.

Lemma 4.2.1 *If $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A_\rho$, the unique polynomial $G^s_{n-1,r}(z; f)$ which minimizes (4.2.2) over all polynomials $Q_{n-1} \in \Pi_{n-1}$, is given by*

$$G^s_{n-1,r}(z; f) = \sum_{j=0}^{n-1} c_j^{(s)} z^j \tag{4.2.4}$$

where

$$c_j^{(s)} = \frac{1}{B_{0,j}(r)} \sum_{\lambda=0}^{\infty} B_{\lambda,j}(r) \Omega_s^\lambda a_{j+\lambda n}, \qquad j = 0, 1, \ldots, n-1$$

and

$$B_{\lambda,j}(r) = \sum_{i=0}^{r-1} (j)_i (j + \lambda n)_i,$$

where $(j)_i = j(j-1), \ldots, (j-i+1)$ and $(j)_0 = 1$.

Before giving the proof of Lemma 4.2.1 we state and prove Lemma 4.2.2.

Let $f_0, f_1, \ldots, f_{r-1}$ be given functions in A_ρ and let $\{p_{\nu,j}\}_{j=0}^{n-1}$ $(\nu = 0, 1, \ldots, r-1)$ be given real numbers. To each set of n numbers $\{p_{\nu,j}\}_{j=0}^{n-1}$ we define an operator $\mathcal{L}_\nu$ on the space of polynomials of degree $n-1$ such that if

$$Q_{n-1}(z) = \sum_{i=0}^{n-1} c_i z^i, \qquad \text{then} \qquad \mathcal{L}_\nu(Q_{n-1}(z)) = \sum_{i=0}^{n-1} c_i p_{\nu,i} z^i.$$

We now first find the polynomial $G^s_{n-1,r}(z; f)$ which minimizes

$$\sum_{\nu=0}^{r-1} \sum_{k=0}^{n-1} |\mathcal{L}_\nu Q_{n-1}(\omega_{s,k}) - f_\nu(\omega_{s,k})|^2, \tag{4.2.5}$$

over all polynomials $Q_{n-1} \in \Pi_{n-1}$. Let the polynomial interpolating $f_\nu(z)$ on $\{\omega_{s,k}\}_{k=0}^{n-1}$ be denoted by $L'_{n-1,s}(z; f_\nu)$. We set

$$L'_{n-1,s}(z; f_\nu) = \sum_{j=0}^{n-1} b_{\nu,j}^{(s)} z^j, \qquad \nu = 0, 1, \ldots, r-1 \tag{4.2.6}$$

where $b_{\nu,j}^{(s)}$ depends upon f_ν and its value on $\{\omega_{s,k}\}_{k=0}^{n-1}$. We shall prove

Lemma 4.2.2 *The unique polynomial $G^s_{n-1,r}(z;f)$ which minimizes (4.2.5) is given by*

$$G^s_{n-1,r}(z;f) = \sum_{j=0}^{n-1} c_j^{(s)} z^j$$

where

$$c_j^{(s)} = \sum_{\nu=0}^{r-1} p_{\nu,j} b_{\nu,j}^{(s)} / \left\{ \sum_{\nu=0}^{r-1} (p_{\nu,j})^2 \right\}, \qquad j = 0, 1, \ldots, n-1. \tag{4.2.7}$$

Proof : Observe that on using (4.2.6), we have

$$\begin{aligned} |\mathcal{L}_\nu Q_{n-1}(\omega_{s,k}) - f_\nu(\omega_{s,k})|^2 &= |\mathcal{L}_\nu Q_{n-1}(\omega_{s,k}) - L'_{n-1,s}(\omega_{s,k};;f_\nu)|^2 \\ &= |\sum_{j=0}^{n-1} c_j p_{\nu,j} \omega^j_{s,k} - \sum_{j=0}^{n-1} b^{(s)}_{\nu,j} \omega^j_{s,k}|^2 \\ &= |\sum_{j=0}^{n-1} (c_j p_{\nu,j} - b^{(s)}_{\nu,j}) \omega^j_{s,k}|^2 \\ &= |\sum_{j=0}^{n-1} d^{(s)}_{\nu,j} \omega^j_{s,k}|^2 \end{aligned}$$

where we have set

$$d^{(s)}_{\nu,j} = b^{(s)}_{\nu,j} - p_{\nu,j} c_j, \qquad 0 \le j \le n-1. \tag{4.2.8}$$

Now for $j \neq pn \ p \geq 0$,

$$\begin{aligned} \sum_{k=0}^{n-1} \omega^j_{s,k} &= \sum_{k=0}^{n-1} e^{\frac{2\pi i}{mn}(km+s)j} \\ &= e^{\frac{2\pi i}{mn}sj} \frac{1 - e^{\frac{2\pi i}{mn} mjn}}{1 - e^{\frac{2\pi i}{mn} mj}} \\ &= e^{\frac{2\pi i}{mn}sj} .0 \\ &= 0. \end{aligned}$$

For $j = pn \ p \geq 0$,

$$\begin{aligned} \sum_{k=0}^{n-1} \omega^j_{s,k} &= \sum_{k=0}^{n-1} e^{\frac{2\pi i}{mn}(km+s)j} \\ &= e^{\frac{2\pi i}{mn}sj} \\ &= e^{\frac{2\pi i}{m}sp} \end{aligned}$$

Thus by using the fact

$$\sum_{k=0}^{n-1} \omega^j_{s,k} = \begin{cases} n e^{\frac{2\pi i}{mn}sj} & \text{if } j = pn, p \geq 0 \\ 0 & \text{if } j \neq pn, p \geq 0 \end{cases} \tag{4.2.9}$$

it follows that

$$\begin{aligned}\sum_{k=0}^{n-1}|\mathcal{L}_\nu Q_{n-1}(\omega_{s,k})-f_\nu(\omega_{s,k})|^2 &= \sum_{k=0}^{n-1}|\sum_{j=0}^{n-1} d_{\nu,j}^{(s)}\omega_{s,k}^j|^2\\ &= \sum_{k=0}^{n-1}(\sum_{j=0}^{n-1} d_{\nu,j}^{(s)}\omega_{s,k}^j)(\sum_{i=0}^{n-1}\overline{d_{\nu,i}^{(s)}\omega_{s,k}^i})\\ &= n\sum_{j=0}^{n-1}|d_{\nu,j}^{(s)}|^2. \end{aligned}\tag{4.2.10}$$

If we put

$$\left.\begin{aligned} c_j &= \rho_j e^{i\theta_j}, \qquad j=0,1,\ldots,n-1\\ b_{\nu,j}^{(s)} &= \sigma_{\nu,j}e^{i\phi_{\nu,j}}, \qquad j=0,1,\ldots,n-1;\ \nu=0,1,\ldots,r-1\end{aligned}\right\}\tag{4.2.11}$$

then from (4.2.8), it follows that

$$\begin{aligned}|d_{\nu,j}^{(s)}|^2 &= |b_{\nu,j}^{(s)}-p_{\nu,j}c_j|^2\\ &= (b_{\nu,j}^{(s)}-p_{\nu,j}c_j)(\overline{b_{\nu,j}^{(s)}}-p_{\nu,j}\overline{c_j})\\ &= (\sigma_{\nu,j}e^{i\phi_{\nu,j}}-p_{\nu,j}\rho_j e^{i\theta_j})(\sigma_{\nu,j}e^{-i\phi_{\nu,j}}-p_{\nu,j}\rho_j e^{-i\theta_j})\\ &= \sigma_{\nu,j}^2-\sigma_{\nu,j}\rho_j p_{\nu,j}e^{i(\phi_{\nu,j}-\theta_j)}\\ &\quad -\sigma_{\nu,j}\rho_j p_{\nu,j}e^{-i(\phi_{\nu,j}-\theta_j)}+\rho_j^2 p_{\nu,j}^2\\ &= p_{\nu,j}^2\rho_j^2+\sigma_{\nu,j}^2-2p_{\nu,j}\rho_j\sigma_{\nu,j}cos(\theta_j-\varphi_{\nu,j}),\ j=0,1,\ldots,n-1\end{aligned}$$

Thus from (4.2.10), the problem (4.2.5) reduces to finding the minimum of the following

$$\sum_{\nu=0}^{r-1}\left\{\sum_{j=0}^{n-1}p_{\nu,j}^2\rho_j^2+\sum_{j=0}^{n-1}\sigma_{\nu,j}^2-2\sum_{j=0}^{n-1}p_{\nu,j}\rho_j\sigma_{\nu,j}cos(\theta_j-\phi_{\nu,j})\right\}\tag{4.2.12}$$

where ρ_j runs over the reals and $0\le\theta_j\le 2\pi$. Differentiating (4.2.12) with respect to ρ_j and θ_j we get the following system of equations to determine ρ_j and θ_j :

$$\left.\begin{aligned} 2\rho_j\textstyle\sum_{\nu=0}^{r-1}(p_{\nu,j})^2-\sum_{\nu=0}^{r-1}2p_{\nu,j}\sigma_{\nu,j}cos(\theta_j-\phi_{\nu,j})=0\\ \textstyle\sum_{\nu=0}^{r-1}p_{\nu,j}\rho_j\sigma_{\nu,j}sin(\theta_j-\phi_{\nu,j})=0,\end{aligned}\right\}$$

or,

$$\left.\begin{aligned} \rho_j\textstyle\sum_{\nu=0}^{r-1}(p_{\nu,j})^2-\sum_{\nu=0}^{r-1}p_{\nu,j}\sigma_{\nu,j}cos(\theta_j-\phi_{\nu,j})=0\\ \textstyle\sum_{\nu=0}^{r-1}p_{\nu,j}\sigma_{\nu,j}sin(\theta_j-\phi_{\nu,j})=0,\end{aligned}\right\}\tag{4.2.13}$$

adding these equations we have

$$\rho_j\sum_{\nu=0}^{r-1}(p_{\nu,j})^2-\sum_{\nu=0}^{r-1}p_{\nu,j}\sigma_{\nu,j}e^{-i(\theta_j-\varphi_{\nu,j})}=0$$

or,

$$\rho_j e^{i\theta_j} \sum_{\nu=0}^{r-1} (p_{\nu,j})^2 - \sum_{\nu=0}^{r-1} p_{\nu,j}\sigma_{\nu,j} e^{i\phi_{\nu,j}} = 0$$

hence,

$$c_j \sum_{\nu=0}^{r-1} (p_{\nu,j})^2 - \sum_{\nu=0}^{r-1} p_{\nu,j} b_{\nu,j}^{(s)} = 0$$

which gives

$$c_j = c_j^{(s)} = \sum_{\nu=0}^{r-1} p_{\nu,j} b_{\nu,j}^{(s)} / (\sum_{\nu=0}^{r-1} (p_{\nu,j})^2),$$

and hence the result.

proof of Lemma 4.2.1 : From (4.2.4) it follows that (4.2.3) reduces to

$$\sum_{\nu=0}^{r-1} \sum_{k=0}^{n-1} | \sum_{j=0}^{n-1} (j)_\nu c_j^{(s)} (\omega_{s,k}^j) - f_\nu(\omega_{s,k})|^2,$$

where $f_\nu(z) = z^\nu f^{(\nu)}(z), \nu = 0, 1, \ldots, r-1$. Thus from Lemma 4.2.2

$$p_{\nu,j} = (j)_\nu, \qquad \nu = 0, 1, \ldots, r-1, \; j = 0, 1, \ldots, n-1.$$

Since $f \in A_\rho$, we have

$$f_\nu(z) = z^\nu f^{(\nu)}(z) = \frac{\nu!}{2\pi i} \int_\Gamma \frac{f(t) z^\nu}{(t-z)^{\nu+1}} dt$$

where Γ is the circle $|t| = R, 1 < R < \rho$. Then

$$f_\nu(\omega_{s,k}) = \frac{\nu!}{2\pi i} \int_\Gamma \frac{f(t)(\omega_{s,k})^\nu}{(t - (\omega_{s,k}))^{\nu+1}} dt.$$

Now since

$$\Pi_{k=0}^{n-1} (z - \omega_{s,k}) = z^n - \Omega_s,$$

where $\Omega_s := (\omega_{s,k})^n$ is a m^{th} root of unity for $k = 0, \ldots, n-1$. Hence if

$$g(\omega_{s,k}) = \frac{\nu!(\omega_{s,k})^\nu}{(t - (\omega_{s,k}))^{\nu+1}},$$

then by Hermite interpolating formula we have

$$L'_{n-1,s}(z,, g) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{\nu! y^\nu}{(t-y)^{\nu+1}} \frac{y^n - z^n}{(y-z)(y^n - \Omega_s)} dy$$

where $\Gamma' : |y| = R', 1 < R' < R$.

$$L'_{n-1,s}(z, g) \;=\; \frac{1}{2\pi i} \int_{\Gamma'} \frac{\nu! y^\nu}{(t-y)^{\nu+1}} \frac{y^n - z^n}{(y-z)(y^n - \Omega_s)} dy$$

$$
\begin{aligned}
&= -residue\left(\frac{\nu! y^\nu}{(t-y)^{\nu+1}}\frac{y^n-z^n}{(y-z)(y^n-\Omega_s)}, y=t\right)\\
&= -\frac{1}{\nu!}\lim_{y\to t}\left(\frac{d^\nu}{dy^\nu}(y-t)^{\nu+1}\frac{\nu! y^\nu}{(t-y)^{\nu+1}}\frac{y^n-z^n}{(y-z)(y^n-\Omega_s)}\right)\\
&= (-1)^{\nu+2}\frac{d^\nu}{dt^\nu}\left(\frac{t^\nu(t^n-z^n)}{(t-z)(t^n-\Omega_s)}\right)\\
&= (-1)^\nu\sum_{k=0}^{n-1}\left(\frac{d^\nu}{dt^\nu}\frac{t^{\nu+n-k-1}z^k}{(t^n-\Omega_s)}\right).
\end{aligned}
$$

Thus,

$$
L'_{n-1,s}(z;f_\nu) = \frac{1}{2\pi i}\int_{\Gamma'} f(t)\left\{(-1)^\nu\sum_{j=0}^{n-1}\frac{d^\nu}{dt^\nu}\left(\frac{t^{n+\nu-j-1}}{t^n-\Omega_s}\right)z^j\right\}dt.
$$

Also

$$
\begin{aligned}
(-1)^\nu\frac{d^\nu}{dt^\nu}\left(\frac{t^{n+\nu-j-1}}{t^n-\Omega_s}\right) &= (-1)^\nu\frac{d^\nu}{dt^\nu}\left(\sum_{\lambda=0}^{\infty}\frac{\Omega_s^\lambda t^{\nu-j-1}}{t^{\lambda n}}\right)\\
&= (-1)^\nu\sum_{\lambda=0}^{\infty}\Omega_s^\lambda\frac{d^\nu}{dt^\nu}t^{-\lambda n+\nu-j-1}\\
&= (-1)^\nu\sum_{\lambda=0}^{\infty}\Omega_s^\lambda(-\lambda n+\nu-j-1)\\
&\quad (-\lambda n+\nu-j-1-1)\ldots\\
&\quad (-\lambda n+\nu-j-1-\nu+1)t^{-\lambda n-j-1}\\
&= \sum_{\lambda=0}^{\infty}\Omega_s^\lambda(\lambda n-\nu+j+1)(\lambda n-\nu+j+2)\ldots\\
&\quad (\lambda n+j)t^{-\lambda n-j-1}\\
&= \sum_{\lambda=0}^{\infty}\Omega_s^\lambda(j+\lambda n)_\nu t^{-\lambda n-j-1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
L'_{n-1,s}(z;f_\nu) &= \frac{1}{2\pi i}\int_{\Gamma'} f(t)\left(\sum_{j=0}^{n-1}\sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu\Omega_s^\lambda t^{-\lambda n-j-1}z^j\right)dt\\
&= \sum_{j=0}^{n-1}\sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu\Omega_s^\lambda z^j\frac{1}{2\pi i}\int_{\Gamma'} f(t)t^{-\lambda n-j-1}dt\\
&= \sum_{j=0}^{n-1}\sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu\Omega_s^\lambda a_{j+\lambda n}z^j.
\end{aligned}
$$

Hence

$$
b_{\nu,j}^{(s)} = \sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu\Omega_s^\lambda a_{j+\lambda n}.
$$

Whence

$$
c_j^{(s)} = \frac{\sum_{\nu=0}^{r-1}(j)_\nu\sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu\Omega_s^\lambda a_{j+\lambda n}}{\sum_{\nu=0}^{r-1}(j)_\nu(j)_\nu}
$$

$$= \frac{1}{B_{0,j}(r)} \sum_{\lambda=0}^{\infty} B_{\lambda,j}(r) a_{j+\lambda n} \Omega_s^{\lambda}, \qquad j = 0, 1, \ldots, n-1,$$

where

$$B_{\lambda,j}(r) = \sum_{i=0}^{r-1} (j)_i (j+\lambda n)_i, \qquad (j)_i = j(j-1), \ldots, (j-i+1).$$

giving the required result.

Thus from (4.2.2)

$$\begin{aligned}
G_{n-1,r}(z;f) &= \frac{1}{m} \sum_{s=0}^{m-1} G_{n-1,r}^{s}(z;f) \\
&= \frac{1}{m} \sum_{s=0}^{m-1} \sum_{j=0}^{n-1} \frac{1}{B_{0,j}(r)} \sum_{\lambda=0}^{\infty} B_{\lambda,j}(r) a_{j+\lambda n} \Omega_s^{\lambda} z^j \\
&= \sum_{j=0}^{n-1} \frac{1}{B_{0,j}(r)} \sum_{\lambda=0}^{\infty} B_{\lambda,j}(r) a_{j+\lambda n} z^j \frac{1}{m} \sum_{s=0}^{m-1} \Omega_s^{\lambda}.
\end{aligned}$$

Now

$$\begin{aligned}
\Omega_s^{\lambda} &= ((\omega_{s,k})^n)^{\lambda} \\
&= e^{\frac{2\pi i}{mn}(km+s)n\lambda} \\
&= e^{\frac{2\pi i}{m} s\lambda}
\end{aligned}$$

hence

$$\frac{1}{m} \sum_{s=0}^{m-1} \Omega_s^{\lambda} = \begin{cases} 1 & \text{if } \lambda = pm \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.14}$$

Thus,

$$G_{n-1,r}(z;f) = \sum_{j=0}^{n-1} \frac{1}{B_{0,j}(r)} \sum_{\lambda=0}^{\infty} B_{\lambda m,j}(r) a_{j+\lambda mn} z^j. \tag{4.2.15}$$

Now for each $\lambda \geq 0$ define

$$G_{n-1,r,\lambda}^{s}(z;f) = \sum_{j=0}^{n-1} \frac{B_{\lambda,j}(r)}{B_{0,j}(r)} a_{j+\lambda n} \Omega_s^{\lambda} z^j, \quad \lambda = 0, 1, \ldots$$

and

$$G_{n-1,r,\lambda}(z;f) = \frac{1}{m} \sum_{s=0}^{m-1} G_{n-1,r,\lambda}^{s}(z;f), \quad \lambda = 0, 1, \ldots.$$

Hence by definition

$$\begin{aligned}
G_{n-1,r,\lambda}(z;f) &= \frac{1}{m} \sum_{s=0}^{m-1} G_{n-1,r,\lambda}^{s}(z;f) \\
&= \frac{1}{m} \sum_{s=0}^{m-1} \sum_{j=0}^{n-1} \frac{B_{\lambda,j}(r)}{B_{0,j}(r)} a_{j+\lambda n} \Omega_s^{\lambda} z^j \\
&= \sum_{j=0}^{n-1} \frac{B_{\lambda,j}(r)}{B_{0,j}(r)} a_{j+\lambda n} z^j \frac{1}{m} \sum_{s=0}^{m-1} \Omega_s^{\lambda}, \qquad (4.2.16)
\end{aligned}$$

which is non-zero only when λ is multiple of m (from (4.2.14)). Thus, for $l \geq 1$ if

$$\Theta_{n-1,r,l,m}(z;f) = G_{n-1,r}(z;f) - \sum_{\lambda=0}^{l-1} G_{n-1,r,\lambda}(z;f)$$

and β is the least positive integer such that $\beta m > l-1$ then,

$$\Theta_{n-1,r,l,m}(z;f) = G_{n-1,r}(z;f) - \sum_{\lambda=0}^{\beta-1} G_{n-1,r,\lambda m}(z;f).$$

Thus, from (4.2.15) and (4.2.16) we have

$$\begin{aligned}
\Theta_{n-1,r,l,m}(z;f) &= G_{n-1,r}(z;f) - \sum_{\lambda=0}^{\beta-1} G_{n-1,r,\lambda m}(z;f) \\
&= \sum_{j=0}^{n-1}\sum_{\lambda=0}^{\infty} \frac{B_{\lambda m,j}(r)}{B_{0,j}(r)} a_{j+\lambda mn} z^j - \\
&\quad \sum_{j=0}^{n-1}\sum_{\lambda=0}^{\beta-1} \frac{B_{\lambda m,j}(r)}{B_{0,j}(r)} a_{j+\lambda mn} z^j \\
&= \sum_{j=0}^{n-1}\sum_{\lambda=\beta}^{\infty} \frac{B_{\lambda m,j}(r)}{B_{0,j}(r)} a_{j+\lambda mn} z^j.
\end{aligned}$$

4.3 In this section we give exact estimates of the sequence $\{\Theta_{n-1,r,l,m}(z;f)\}$ and study its pointwise behaviour. If we set

$$q_{\beta,m}(R) = \overline{\lim_{n\to\infty}} \max_{|z|=R} |\Theta_{n-1,r,l,m}(z;f)|^{1/n}$$

and

$$K_{\beta,m}(R,\rho) = \begin{cases} \frac{R}{\rho^{1+\beta m}}, & \text{if} \quad R \geq \rho \\ \frac{1}{\rho^{\beta m}} & \text{if} \quad 0 < R < \rho \end{cases}$$

Then,

Theorem 4.3.1 *If $f \in A_\rho$, l is a positive integer and β is the least positive integer such that $\beta m > l-1$ and $R > 0$ then*

$$q_{\beta,m}(R) = K_{\beta,m}(R,\rho).$$

Proof : Since $f \in A_\rho$, we have

$$a_k = \mathcal{O}(\rho-\epsilon)^{-k} \tag{4.3.1}$$

for every ϵ satisfying $0 < \epsilon < \rho - 1$ and $k \geq k_0(\epsilon)$. Let R be fixed, $| z |= R$ and if $R < \rho$ then we assume $\epsilon > 0$ so small that $R < \rho - \epsilon$ be satisfied as well. Then by the definition of $\Theta_{n-1,r,l,m}(z; f)$ and Lemma 3.2.1 for $q = mn$ and $l = \beta$ we obtain

$$\begin{aligned}
\Theta_{n-1,r,l,m}(z;f) &= \sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k \\
&= \mathcal{O}\left(\sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}\frac{| z |^k}{(\rho-\epsilon)^{k+jmn}}\right) \\
&= \mathcal{O}\left(\sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1}\frac{\sum_{i=0}^{r-1}(k)_i(k+jmn)_i}{\sum_{i=0}^{r-1}(k)_i(k)_i}\frac{| z |^k}{(\rho-\epsilon)^{k+jmn}}\right) \\
&= \mathcal{O}\left(\sum_{k=0}^{n-1}\sum_{i=0}^{r-1}(k)_i|z|^k\sum_{j=\beta}^{\infty}\frac{(k+jmn)_i}{(\rho-\epsilon)^{k+jmn}}\right) \\
&= \mathcal{O}\left(\sum_{k=0}^{n-1}\sum_{i=0}^{r-1}(k)_i|z|^k S_{mn,\beta,i}(\rho-\epsilon)\right) \\
&= \mathcal{O}\left(\sum_{k=0}^{n-1}\sum_{i=0}^{r-1}(k)_i|z|^k(k+\beta mn)_i(\rho-\epsilon)^{-\beta mn-k}\right) \\
&= \mathcal{O}\left((\rho-\epsilon)^{-\beta mn}\sum_{k=0}^{n-1}\sum_{i=0}^{r-1}(k)_i(k+\beta mn)_i\frac{R^k}{(\rho-\epsilon)^{-k}}\right) \\
&= \mathcal{O}\begin{cases} N(n)\frac{R^n}{(\rho-\epsilon)^{\beta mn+n}} & \text{for} \quad R \geq \rho \\ N(n)\frac{1}{(\rho-\epsilon)^{\beta mn}} & \text{for} \quad 0 < R < \rho, \end{cases}
\end{aligned}$$

where $N(n)$ is a quantity dependent on n with $\lim_{n\to\infty}(N(n))^{1/n} = 1$. Thus

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Theta_{n-1,r,l,m}(z;f)|^{1/n} &\leq \frac{R}{(\rho-\epsilon)^{1+\beta m}}, \qquad \text{if} \quad R \geq \rho \\
&\leq \frac{1}{(\rho-\epsilon)^{\beta m}} \qquad \text{if} \quad 0 < R < \rho.
\end{aligned}$$

Being $\epsilon > 0$ arbitrary small this gives

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Theta_{n-1,r,l,m}(z;f)|^{1/n} &\leq \frac{R}{\rho^{1+\beta m}}, \qquad \text{if} \quad R \geq \rho \\
&\leq \frac{1}{\rho^{\beta m}} \qquad \text{if} \quad 0 < R < \rho.
\end{aligned}$$

To prove the opposite inequality let first $R \geq \rho$, then

$$\begin{aligned}
\Theta_{n-1,r,l,m}(z;f) &= \sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k \\
&= \sum_{k=0}^{n-\beta m-2}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + \sum_{k=n-\beta m-1}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + \\
&\quad + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k.
\end{aligned}$$

Thus,

$$\sum_{k=n-\beta m-1}^{n-1} \frac{B_{\beta m,k}}{B_{0,k}} a_{k+\beta mn} z^k = \Theta_{n-1,r,l,m}(z;f) - \sum_{k=0}^{n-\beta m-2} \frac{B_{\beta m,k}}{B_{0,k}} a_{k+\beta mn} z^k - \\ - \sum_{j=\beta+1}^{\infty} \sum_{k=0}^{n-1} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} z^k$$

gives, by Cauchy integral formula, for $n-\beta m-1 \le k \le n-1$,

$$\frac{B_{\beta m,k}}{B_{0,k}} a_{k+\beta mn} = \frac{1}{2\pi i} \int_{|z|=R} \frac{\Theta_{n-1,r,l,m}(z;f)}{z^{k+1}} dz - \\ - \frac{1}{2\pi i} \sum_{k'=0}^{n-\beta m-2} \frac{B_{\beta m,k'}}{B_{0,k'}} a_{k'+\beta mn} \int_{|z|=R} \frac{z^{k'}}{z^{k+1}} dz \\ - \frac{1}{2\pi i} \int_{|z|=R} \frac{\sum_{j=\beta+1}^{\infty} \sum_{k'=0}^{n-1} \frac{B_{jm,k'}}{B_{0,k'}} a_{k'+jmn} z^{k'}}{z^{k+1}} dz.$$

Since $\int_{|z|=R} \frac{z^{k'}}{z^{k+1}} dz$ is non zero only for $k=k'$, the middle integral on the right hand side in above equation is zero. Then by the definition of $q_{\beta,m}(R)$ and (4.3.1) we have for every $n \ge n_0(\epsilon)$ and a constant M, which need not be same at each occurrence

$$\left| \frac{B_{\beta m,k}}{B_{0,k}} a_{k+\beta mn} \right| \le M \frac{(q_{\beta,m}(R)+\epsilon)^n}{R^k} + \mathcal{O}\left(N(n) \frac{R^n}{R^k (\rho-\epsilon)^{n+(\beta+1)mn}} \right) \\ \le M \frac{(q_{\beta,m}(R)+\epsilon)^n}{R^k} + \mathcal{O}\left(N(n) \frac{1}{(\rho-\epsilon)^{n(1+m(\beta+1))}} \right).$$

Let $\epsilon > 0$ be so small that

$$(\rho-\epsilon)^{-(1+m(\beta+1))} < \rho^{-(1+\beta m)}.$$

Thus,

$$(q_{\beta,m}(R)+\epsilon)^n \ge \frac{R^k}{M} \left(\left| \frac{B_{\beta m,k}}{B_{0,k}} a_{k+\beta mn} \right| - \mathcal{O}\left(\frac{N(n)}{\rho^{n(1+\beta m)}} \right) \right)$$

hence,

$$q_{\beta,m}(R)+\epsilon \ge \overline{\lim_{n\to\infty}} \left\{ |a_{k+\beta mn}|^{\frac{1}{k+\beta mn}} \right\}^{\frac{k+\beta mn}{n}} \left\{ \frac{B_{\beta m,k}}{B_{0,k}} \frac{R^k}{M} \right\}^{\frac{1}{n}}.$$

Now since $n-\beta m-1 \le k \le n-1$ we have, $\lim_{n\to\infty} \frac{k}{n} = 1$ and so

$$q_{\beta,m}(R)+\epsilon \ge \frac{R}{\rho^{1+\beta m}}.$$

Since ϵ is arbitrary, this yeilds

$$q_{\beta,m}(R) \ge \frac{R}{\rho^{1+\beta m}} \qquad \text{for} \qquad R \ge \rho.$$

For the case $0 < R < \rho$, we write

$$\begin{aligned}
\Theta_{n-1,r,l,m}(z;f) &= \sum_{j=l}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k \\
&= \sum_{k=0}^{\beta m-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + \sum_{k=\beta m}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + \\
&\quad \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k
\end{aligned}$$

whence,

$$\begin{aligned}
\sum_{k=0}^{\beta m-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k &= \Theta_{n-1,r,l,m}(z;f) - \sum_{k=\beta m}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k - \\
&\quad - \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k.
\end{aligned}$$

By Cauchy integral formula we have, for $0 \le k \le \beta m - 1$,

$$\begin{aligned}
\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn} &= \frac{1}{2\pi i}\int_{|z|=R}\frac{\Theta_{n-1,r,l,m}(z;f)}{z^{k+1}}dz - \\
&\quad -\frac{1}{2\pi i}\sum_{k'=\beta m}^{n-1}\frac{B_{\beta m,k'}}{B_{0,k'}}a_{k'+\beta mn}\int_{|z|=R}\frac{z^{k'}}{z^{k+1}}dz \\
&\quad -\frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{j=\beta+1}^{\infty}\sum_{k'=0}^{n-1}\frac{B_{jm,k'}}{B_{0,k'}}a_{k'+jmn}z^{k'}}{z^{k+1}}dz.
\end{aligned}$$

Using the same arguments as earlier, we then have,

$$\begin{aligned}
\left|\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}\right| &\le M\frac{(q_{\beta,m}(R)+\epsilon)^n}{R^k} + \mathcal{O}\left(N(n)\frac{1}{R^k(\rho-\epsilon)^{mn(\beta+1)}}\right) \\
&\le M(q_{\beta,m}(R)+\epsilon)^n + \mathcal{O}\left(N(n)\frac{1}{(\rho-\epsilon)^{mn(\beta+1)}}\right).
\end{aligned}$$

Let $\epsilon > 0$ be so small that

$$(\rho-\epsilon)^{-(\beta+1)} < \rho^{-\beta},$$

then,

$$(q_{\beta,m}(R)+\epsilon)^n \ge \frac{1}{M}\left(\left|\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}\right| - \mathcal{O}\left(N(n)\frac{1}{\rho^{\beta mn}}\right)\right)$$

or,

$$\begin{aligned}
q_{\beta,m}(R)+\epsilon &\ge \overline{\lim_{n\to\infty}}\left\{|a_{k+\beta mn}|^{\frac{1}{k+\beta mn}}\right\}^{\frac{k+\beta mn}{n}}\left\{\left|\frac{B_{\beta m,k}}{MB_{0,k}}\right|\right\}^{\frac{1}{n}} \\
&= \frac{1}{\rho^{\beta m}}.
\end{aligned}$$

Since ϵ is arbitrary, this gives

$$q_{\beta,m}(R) \geq \frac{1}{\rho^{\beta m}} \qquad \text{for} \qquad 0 < R < \rho$$

which completes the proof.

Since

$$\Theta_{n-1,r,l,m}(z;f) = \sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} z^k,$$

for $R = 0$, that is $z = 0$

$$\Theta_{n-1,r,l,m}(z;f) = \sum_{j=\beta}^{\infty} \frac{B_{jm,0}}{B_{0,0}} a_{jmn}.$$

Now $(k)_0 = 1$, thus by the definition of $B_{j,k}$, $B_{jm,0} = 1$ and $B_{0,0} = 1$. Thus,

$$\Theta_{n-1,r,l,m}(0;f) = \sum_{j=\beta}^{\infty} a_{jmn}.$$

Consider the function

$$\begin{aligned} F(z) &= \frac{1}{1-(z/\rho)^{(\beta+1)m}} \\ &= \sum_{n=0}^{\infty}(\frac{z}{\rho})^{(\beta+1)mn}. \end{aligned}$$

Note that for $F(z)$, $a_{\beta mn} = 0$. Which gives

$$\begin{aligned} \Theta_{n-1,r,l,m}(0;F) &= \sum_{j=\beta+1}^{\infty} a_{jmn} \\ &= \mathcal{O}\left(\frac{1}{\rho^{(\beta+1)mn}}\right). \end{aligned}$$

Hence for $R = 0$

$$q_{\beta,m}(R) \leq \frac{1}{\rho^{(\beta+1)m}} < \frac{1}{\rho^{\beta m}}.$$

Whence

Remark 4.3.1 For $R = 0$ Theorem 4.3.1 does not hold.

Further for $r = 1$, $B_{j,k}(r) = 1$ hence

Remark 4.3.2 For $r = 1$ Theorem 4.3.1 reduces to Theorem 2.2.1.

Corollary 4.3.1 *If $l \geq 1$, f is analytic in an open domain containing $|z| \leq 1$ and $q_{\beta,m}(R) = K_{\beta,m}(R,\rho)$ for some $R > 0, \rho > 1$ then $f \in A_\rho$.*

Proof Given that f is analytic in an open domain containing $|z| \leq 1$. Hence $f \in A_{\rho'}$ for some $\rho' > 1$. Thus by Theorem 4.2.1 $q_{\beta,m}(R) = K_{\beta,m}(R, \rho')$, and from the hypothesis $q_{\beta,m}(R) = K_{\beta,m}(R, \rho)$. That is $K_{\beta,m}(R, \rho') = K_{\beta,m}(R, \rho)$ and hence $\rho' = \rho$ which gives $f \in A_\rho$.

Next, we consider the pointwise behavior of $\Theta_{n-1,r,l,m}(z; f)$. We shall prove not only that the sequence $\Theta_{n-1,r,l,m}(z; f)$ is bounded at most at βm points in $|z| > \rho^{1+\beta m}$ but

Theorem 4.3.2 *Let $f \in A_\rho, \rho > 1$, $l \geq 1$ and β is the least positive integer such that $\beta m > l - 1$. Then*

$$(i) \qquad \overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z; f) \mid^{1/n} = \frac{\mid z \mid}{\rho^{1+\beta m}}$$

for all but at most βm distinct points in $\mid z \mid > \rho$.

$$(ii) \qquad \overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z; f) \mid^{1/n} = \frac{1}{\rho^{\beta m}}$$

for all but at most $\beta m - 1$ distinct points in $0 < \mid z \mid < \rho$.

Proof : Let first $|z| = R > \rho$. Consider

$$\Gamma_{n-1,r,l,m}(z; f) = \sum_{j=\beta}^{\infty} \sum_{k=0}^{n-1} \frac{B_{jm,k+\beta m}}{B_{0,k+\beta m}} a_{k+jmn} z^k. \tag{4.3.2}$$

Now since $f \in A_\rho$ so

$$\sum_{j=\beta}^{\infty} \frac{B_{jm,k+\beta m}}{B_{0,k+\beta l}} a_{k+jmn} = \mathcal{O}\left(N(n)(\rho - \epsilon)^{-(k+\beta mn)}\right).$$

thus,

$$\begin{aligned} \overline{\lim_{k\to\infty}} \Big| \sum_{j=\beta}^{\infty} \frac{B_{jm,k+\beta m}}{B_{0,k+\beta m}} a_{k+jmn} \Big|^{1/k} &\leq \overline{\lim_{k\to\infty}} \left(KN(n)(\rho - \epsilon)^{-(k+\beta mn)}\right)^{1/k} \\ &\leq (\rho - \epsilon)^{-1} \\ &< 1. \end{aligned}$$

Thus sequence (4.3.2) is convergent. Also from the expression of $\Theta_{n-1,r,l,m}(z; f)$ and $\Gamma_{n-1,r,l,m}(z; f)$ it is clear that

$$\overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z; f) \mid^{1/n} = \overline{\lim_{n\to\infty}} \mid \Gamma_{n-1,r,l,m}(z; f) \mid^{1/n} . \tag{4.3.3}$$

Thus for $|z| = R > \rho$,

$$\begin{aligned}
h(z) &= \Theta_{n-1,r,l,m}(z;f) - z^{\beta m}\Gamma_{n,r,l,m}(z;f) \\
&= \sum_{j=\beta}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k - z^{\beta m}\sum_{j=\beta}^{\infty}\sum_{k=0}^{n}\frac{B_{jm,k+ml}}{B_{0,k+ml}}a_{k+jm(n+1)}z^k \\
&= \sum_{k=0}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k - \\
&\quad -z^{\beta m}\sum_{k=0}^{n}\frac{B_{l,k+\beta m}}{B_{0,k+\beta m}}a_{k+\beta m(n+1)}z^k - z^{\beta m}\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n}\frac{B_{jm,k+\beta m}}{B_{0,k+\beta l}}a_{k+jm(n+1)}z^k \\
&= \sum_{k=0}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k - \sum_{k=\beta m}^{n+\beta m}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta nm}z^k + \\
&\quad + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k - z^{\beta m}\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n}\frac{B_{jm,k+\beta m}}{B_{0,k+\beta m}}a_{k+jm(n+1)}z^k \\
&= \sum_{k=0}^{\beta m-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + \sum_{k=\beta m}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k - \\
&\quad - \sum_{k=\beta m}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k - \sum_{k=n}^{n+\beta m}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + \\
&\quad + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k - z^{\beta m}\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n}\frac{B_{jm,k+\beta l}}{B_{0,k+\beta m}}a_{k+\beta m(n+1)}z^k \\
&= \sum_{k=0}^{\beta m-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k - \sum_{k=0}^{\beta m}\frac{B_{\beta m,k+n}}{B_{0,k+n}}a_{k+\beta mn+n}z^{k+n} + \\
&\quad + \sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}z^k - z^{\beta m}\sum_{j=\beta+1}^{\infty}\sum_{k=0}^{n}\frac{B_{jm,k+\beta m}}{B_{0,k+\beta m}}a_{k+jm(n+1)}z^k \qquad (4.3.4) \\
&= -\sum_{k=0}^{\beta m}\frac{B_{\beta m,k+n}}{B_{0,k+n}}a_{k+(1+\beta m)n}z^{k+n} + ON(n)\left(\frac{|z|^{\beta m}}{(\rho-\epsilon)^{(n+1)\beta m}} + \right. \\
&\quad \left. + \frac{|z|^n}{(\rho-\epsilon)^{(1+(\beta+1)m)n}}\right) + O\left(\frac{|z|^n}{(\rho-\epsilon)^{n(1+(\beta+1)m)+(\beta+1)m}}\right) \\
&= -\sum_{k=0}^{\beta m}\frac{B_{\beta m,k+n}}{B_{0,k+n}}a_{k+n(1+\beta m)}z^{k+n} + \\
&\quad + ON(n)\left(\frac{1}{(\rho-\epsilon)^{n\beta m}} + \frac{|z|^n}{(\rho-\epsilon)^{(1+(\beta+1)m)n}}\right). \qquad (4.3.5)
\end{aligned}$$

Thus from (4.3.5), (2.3.8) and (2.3.9) we have

$$h(z) = -\sum_{k=0}^{\beta m}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+n(1+\beta m)}z^{k+n} + ON(n)\left(\frac{|z|}{\rho^{1+\beta m}} - \eta\right)^n \qquad (4.3.6)$$

where η is a positive number .

If we assume that in (i) equality does not hold at more than βm points say $\beta m + 1$ points

then

$$\overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z_j; f) \mid^{1/n} < \frac{\mid z_j \mid}{\rho^{1+\beta m}} \qquad , j = 1, 2, \cdots \beta m + 1$$

for $z_1, z_2, \cdots, z_{\beta m+1}$ with $\mid z_1 \mid, \mid z_2 \mid, \cdots, \mid z_{\beta m+1} \mid > \rho$.

Let

$$\overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z_j; f) \mid^{1/n} = \frac{\mid z_j \mid}{\rho^{1+\beta m}} - s \qquad \text{for some } s > 0$$

that is

$$\mid \Theta_{n-1,r,l,m}(z_j; f) \mid \leq \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - s + \epsilon \right)^n \qquad \forall n > n_0(\epsilon).$$

Hence from (4.3.3) we have also that

$$\mid \Gamma_{n-1,r,l,m}(z_j; f) \mid \leq \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - s + \epsilon \right)^{n+1} \qquad \forall n > n_0(\epsilon)$$

therefore,

$$\begin{aligned}
\mid h(z_j) \mid &= \mid \Theta_{n-1,r,l,m}(z_j; f) - z_j^{\beta m} \Gamma_{n-1,r,l,m}(z_j; f) \mid \\
&\leq \mid \Theta_{n-1,r,l,m}(z_j; f) \mid + \mid z_j^{\beta m} \Gamma_{n-1,r,l,m}(z_j; f) \mid \\
&\leq \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - s + \epsilon \right)^n + \mid z_j \mid^{\beta m} \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - s + \epsilon \right)^{n+1} \\
&= \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - s + \epsilon \right)^n \left(1 + \mid z_j \mid^{\beta m} \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - s + \epsilon \right) \right) \qquad (4.3.7)
\end{aligned}$$

hence

$$\overline{\lim_{n\to\infty}} \mid h(z_j) \mid^{1/n} \leq \frac{\mid z_j \mid}{\rho^{1+\beta m}} - s \ , \qquad r > 0$$

that is

$$\overline{\lim_{n\to\infty}} \mid h(z_j) \mid^{1/n} < \frac{\mid z_j \mid}{\rho^{1+\beta m}}, \qquad j = 1, 2, \cdots, \beta m + 1.$$

Now from (4.3.6)

$$\begin{aligned}
\sum_{k=0}^{\beta m} \frac{B_{\beta m,k+n}}{B_{0,k+n}} a_{k+n(1+\beta m)} z_j^{k+n} &= ON(n) \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - \eta \right)^n - h(z_j) \\
&= \delta_{j,n} \ (say) \qquad (4.3.8)
\end{aligned}$$

where from (4.3.7) for sufficiently large n and constant $k > 1$

$$\begin{aligned}
\mid \delta_{j,n} \mid &\leq ON(n) \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - \eta \right)^n + kN(n) \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - s \right)^n \\
&= k_1 N(n) \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - \eta_1 \right)^n \qquad (4.3.9)
\end{aligned}$$

$$\text{for } k_1 > 1,\ \eta_1 > 0,\ j = 1, 2, \cdots, \beta m + 1.$$

From (4.3.8) we have

$$\sum_{k=0}^{\beta m} \frac{B_{\beta m,k+n}}{B_{0,k+n}} a_{k+n(1+\beta m)} z_j^k = z_j^{-n} \delta_{j,n} \tag{4.3.10}$$

where $|\ \delta_{j,n}\ | \leq k_1 N(n) \left(\frac{|z_j|}{\rho^{1+\beta m}} - \eta_1\right)^n$ for suficiently large $n, k_1 > 1, \eta_1 > 0$ and $1 \leq j \leq \beta m + 1$.

Solving system of equations (4.3.10) we have . Thus,

$$\frac{B_{\beta m,k+n}}{B_{0,k+n}} a_{(\beta m+1)n+k} = \sum_{j=1}^{\beta m+1} c_j^{(k)} z_j^{-n} \delta_{j,n}\ ,\ \ 0 \leq k \leq \beta m$$

where matrix $(c_j^{(k)})$, $1 \leq j \leq \beta m + 1,\ 0 \leq k \leq \beta m$ is inverse of coefficint matrix $(z_j^k), 1 \leq j \leq \beta m + 1$, $0 \leq k \leq \beta m$ hence $c_j^{(k)}$ are independent of n by which, from (4.3.9) we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \mid a_{(\beta m+1)n+k} \mid^{1/(\beta m+1)n+k} \\
&= \overline{\lim_{n\to\infty}} \left(\sum_{j=1}^{\beta m+1} c_j^{(k)} z_j^{-n} k_1 N(n) \left(\frac{\mid z_j \mid}{\rho^{1+\beta m}} - \eta_1 \right)^n \right)^{1/(\beta m+1)n+k} \\
&= \overline{\lim_{n\to\infty}} \left(\sum_{j=1}^{\beta m+1} c_j^{(k)} k_1 N(n) \left(\frac{1}{\rho^{\beta m+1}} - \frac{\eta_1}{\mid z_j \mid} \right)^n \right)^{1/(\beta m+1)n+k} \\
&\leq \overline{\lim_{n\to\infty}} \left(k_2 N(n) \left(\frac{1}{\rho^{\beta m+1}} - \frac{\eta_1}{\mid z_j \mid} \right)^n \right)^{1/(\beta m+1)n+k}
\end{aligned} \tag{4.3.11}$$

where $$k_2 = k_1(\beta m + 1) \sum_{j=1}^{\beta m+1} c_j^{(k)} > 1\ ,\ 0 \leq k \leq \beta m$$

hence from (4.3.11) we have

$$\overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} < \frac{1}{\rho}$$

which contradicts that $f \in A_\rho$. Hence our assumption that in (i) equality does not hold at more than βm points was wrong and thus

$$\overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z; f) \mid^{1/n} = \frac{\mid z \mid}{\rho^{1+\beta m}}$$

for all but at most βm distinct points in $\mid z \mid > \rho$.

In the proof of (ii), one can argue similarly using (4.3.4)

$$\begin{aligned}
h(z) \ = \ & \sum_{k=0}^{\beta m-1} \frac{B_{\beta m,k}}{B_{0,k}} a_{k+\beta mn} z^k - \sum_{k=0}^{\beta m} \frac{B_{\beta m,k+n}}{B_{0,k+n}} a_{k+n(1+\beta m)} z^k + \\
& \sum_{j=\beta+1}^{\infty} \sum_{k=0}^{n-1} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} z^k - z^{\beta m} \sum_{j=\beta+1}^{\infty} \sum_{k=0}^{n} \frac{B_{jm,k+\beta m}}{B_{0,k+\beta m}} a_{k+jm(n+1)} z^k
\end{aligned}$$

for $|z|<\rho$,

$$\begin{aligned} h(z) &= \sum_{k=0}^{\beta m-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + ON(n)\left(\frac{|z|^n}{(\rho-\epsilon)^{(\beta m+1)n}}+\right. \\ &\left. + \frac{1}{(\rho-\epsilon)^{n(\beta+1)m}} + \frac{1}{(\rho-\epsilon)^{(n+1)(\beta+1)m}}\right) \\ &= \sum_{k=0}^{\beta m-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k + ON(n)\left(\frac{|z|^n}{(\rho-\epsilon)^{(\beta m+1)n}} + \frac{1}{(\rho-\epsilon)^{n(\beta+1)m}}\right). \end{aligned}$$

This together with (2.3.21) and (2.3.22) we have

$$h(z) = \sum_{k=0}^{\beta m-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+n\beta m}z^k + ON(n)\left(\frac{1}{\rho^{\beta m}}-\eta\right)^n \tag{4.3.12}$$

where η is a positive number.

If we assume that in (ii) equality does not hold at more than $\beta m-1$ points say βm points then

$$\overline{\lim_{n\to\infty}}\ |\Theta_{n-1,r,l,m}(z_j;f)|^{1/n} < \frac{1}{\rho^{\beta m}} \qquad j=1,2,\cdots,\beta m$$

for $z_1, z_2, \cdots, z_{\beta m}$ with $|z_1|, |z_2|, \cdots, |z_{\beta m}| < \rho$. By the similar arguments as for the case $|z|>\rho$

$$\overline{\lim_{n\to\infty}}\ |h(z_j)|^{1/n} < \frac{1}{\rho^{\beta m}} \qquad j=1,2,\cdots,\beta m$$

Now from (4.3.12)

$$\begin{aligned} \sum_{k=0}^{\beta m-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+n\beta m}z_j^k &= ON(n)\left(\frac{1}{\rho^{\beta m}}-\eta\right)^n - h(z_j) \\ &= \delta_{j,n} \qquad (say) \end{aligned} \tag{4.3.13}$$

where, as for case (i)

$$|\delta_{j,n}| \le k_1 N(n)\left(\frac{1}{\rho^{\beta m}}-\eta_1\right)^n \tag{4.3.14}$$

for sufficiently large $n, k_1 > 1, \eta_1 > 0$ and $1 \le j \le \beta m$. Solving this system of equations (4.3.13) as earlier

$$\frac{B_{\beta m,k}}{B_{0,k}}a_{k+n\beta m} = \sum_{j=1}^{\beta m} c_j^{(k)}\delta_{j,n}$$

where c_j^k are appropiate constants independent of n. Hence from (4.3.14)

$$\begin{aligned} &\overline{\lim_{n\to\infty}}\ |a_{k+n\beta m}|^{1/k+n\beta m} \\ &\le \overline{\lim_{n\to\infty}}\left(\sum_{j=1}^{\beta m} c_j^{(k)}k_1 N(n)\left(\frac{1}{\rho^{\beta m}}-\eta_1\right)^n\right)^{1/k+n\beta m} \end{aligned}$$

$$= \overline{\lim_{n\to\infty}} \left(k_2 N(n) \left(\frac{1}{\rho^{\beta m}} - \eta_1 \right)^n \right)^{1/k+n\beta m}$$

where $k_2 = \sum_{j=1}^{\beta m} c_j^{(k)} k_1 > 1 \qquad , 0 \le k \le \beta m - 1$

thus

$$\overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} < \frac{1}{\rho}$$

which contradicts that $f \in A_\rho$. Hence

$$\overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z; f) \mid^{1/n} = \frac{1}{\rho^{\beta m}}$$

for all but at most $\beta m - 1$ distinct points in $0 <\mid z \mid< \rho$.

Remark 4.3.3 For $r = 1$ Theorem 4.3.2 reduces to Theorem 2.3.2.

From Theorem 4.3.1 and Theorem 4.3.2 we have

$$\overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z; f) \mid^{1/n} < \frac{\mid z \mid}{\rho^{1+\beta m}}$$

for at most βm distinct points in $\mid z \mid > \rho$. That is in $|z| > \rho^{1+\beta m}$

$$\overline{\lim_{n\to\infty}} \mid \Theta_{n-1,r,l,m}(z; f) \mid^{1/n} < B, \;\; B > 1$$

for at most βm distinct points. In other words we can say that

Remark 4.3.4 Let $f \in A_\rho, \rho > 1$ and $l \ge 1$ with β the smallest positive integer such that $\beta m > l - 1$ then the sequence $\{\Theta_{n-1,r,l,m}(z; f)\}_{n=1}^\infty$ can be bounded at most at βm distinct points in $|z| > \rho^{1+\beta m}$.

Corollary 4.3.2 *If f is analytic on $|z| \le 1$ and if $\Theta_{n-1,r,l,m}(z; f)$ is uniformly bounded in every closed subdomain of $|z| < \rho^{1+\beta m}$ then f is analytic in $|z| < \rho$.*

Proof If f is analytic on $|z| \le 1$. Let $f \in A_{\rho_1}$, then from Theorem 4.3.1, $q_{\beta,m} = K_{\beta,m}(R, \rho_1)$. Thus, by above Remark 4.3.4 $\{\Theta_{n-1,r,l,m}(z; f)\}_{n=1}^\infty$ can be bounded at most at βm distinct points in $|z| > {\rho_1}^{1+\beta m}$. Also it is given that $\Theta_{n-1,r,l,m}(z; f)$ is uniformly bounded in every closed subdomain of $|z| < \rho^{1+\beta m}$. Hence $\rho_1 < \rho$ is not possible. That is $\rho_1 \ge \rho$ which gives that f is analytic in $|z| < \rho$.

4.4 The object of this note is to consider roots of α^{mn} in place of roots of unity, where $|\alpha| < \rho$. That is to study the polynomials $G_{n-1,r}(z, \alpha; f)$, where

$$G_{n-1,r}(z, \alpha; f) = \frac{1}{m} \sum_{k=0}^{m-1} G_{n-1,r}^s(z, \alpha; f) \tag{4.4.1}$$

where for each $s = 0, \ldots, m-1$, $G^s_{n-1,r}(z,\alpha; f)$ is a polynomial of degree $n-1$ which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{n-1} |Q^{(\nu)}_{n-1}(\phi_{s,k}) - f^{(\nu)}(\phi_{s,k})|^2 \tag{4.4.2}$$

where $(\phi_{s,k})^{mn} = \alpha^{mn}$. That is

$$\phi_{s,k} = \alpha e^{\frac{2\pi i}{mn}(km+s)} = \alpha\omega_{s,k}, \;\; k = 0, \ldots, n-1 \;\; s = 0, \ldots, m-1$$

where $\omega_{s,k}$ are given by (4.2.1). Hence (4.4.2) can be replaced by

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{n-1} |Q^{(\nu)}_{n-1}(\alpha\omega_{s,k}) - f^{(\nu)}(\alpha\omega_{s,k})|^2 \tag{4.4.3}$$

Lemma 4.4.1 *If $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A_\rho$, the unique polynomial $G^s_{n-1,r}(z,\alpha; f)$ which minimizes (4.4.3) over all polynomials $Q_{n-1} \in \Pi_{n-1}$, is given by*

$$G^s_{n-1,r}(z,\alpha; f) = \sum_{j=0}^{n-1} c_j^{(s)}(\alpha) z^j \tag{4.4.4}$$

where

$$c_j^{(s)}(\alpha) = \frac{1}{B_{0,j}(r)} \sum_{\lambda=0}^{\infty} B_{\lambda,j}(r) a_{j+\lambda n} \alpha^{\lambda n} \Omega_s^\lambda, \qquad j = 0, 1, \ldots, n-1 \tag{4.4.5}$$

and

$$B_{\lambda,j}(r) = \sum_{i=0}^{r-1} (j)_i (j+\lambda n)_i,$$

where $(j)_i = j(j-1), \ldots, (j-i+1)$ and $(j)_0 = 1$.

Before giving the proof of Lemma 4.4.1 we state and prove Lemma 4.4.2.

Let $f_0, f_1, \ldots, f_{r-1}$ be given functions in A_ρ and let $\{p_{\nu,j}\}_{j=0}^{n-1}$ $(\nu = 0, 1, \ldots, r-1)$ be given real numbers. To each set of n numbers $\{p_{\nu,j}\}_{j=0}^{n-1}$ we define an operator $\mathcal{L}_\nu$ on the space of polynomials of degree $n-1$ such that if

$$Q_{n-1}(z) = \sum_{i=0}^{n-1} c_i z^i, \qquad \text{then} \qquad \mathcal{L}_\nu(Q_{n-1}(z)) = \sum_{i=0}^{n-1} c_i p_{\nu,i} z^i.$$

We now first find the polynomial $G^s_{n-1,r}(z,\alpha; f)$ which minimizes

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{n-1} |\mathcal{L}_\nu Q_{n-1}(\alpha\omega_{s,k}) - f_\nu(\alpha\omega_{s,k})|^2, \tag{4.4.6}$$

over all polynomials $Q_{n-1} \in \Pi_{n-1}$. Let the polynomial interpolating $f_\nu(z)$ on $\{\alpha\omega_{s,k}\}_{k=0}^{n-1}$ be denoted by $L'_{n-1,s}(z,\alpha; f_\nu)$. We set

$$L'_{n-1,s}(z,\alpha; f_\nu) = \sum_{j=0}^{n-1} b^{(s)}_{\nu,j,\alpha} z^j, \qquad \nu = 0, 1, \ldots, r-1 \tag{4.4.7}$$

where $b^{(s)}_{\nu,j,\alpha}$ depends upon f_ν and its value on $\{\alpha\omega_{s,k}\}_{k=0}^{n-1}$. We shall prove

Lemma 4.4.2 *The unique polynomial $G^s_{n-1,r}(z,\alpha;f)$ which minimizes (4.4.6) is given by*

$$G^s_{n-1,r}(z,\alpha;f)=\sum_{j=0}^{n-1}c_j^{(s)}(\alpha)z^j$$

where

$$c_j^{(s)}(\alpha)=\sum_{\nu=0}^{r-1}p_{\nu,j}b^{(s)}_{\nu,j,\alpha}\Big/\left\{\sum_{\nu=0}^{r-1}(p_{\nu,j})^2\right\},\qquad j=0,1,\ldots,n-1. \tag{4.4.8}$$

Proof : The proof here is analogous to the proof of Lemma 4.2.2. Thus we limit to the sketch of the proof where it differ from the earlier one. Observe that on using (4.4.7), we have

$$\begin{aligned}|\mathcal{L}_\nu Q_{n-1}(\alpha\omega_{s,k})-f_\nu(\alpha\omega_{s,k})|^2 &= |\mathcal{L}_\nu Q_{n-1}(\alpha\omega_{s,k})-L'_{n-1,s}(\alpha\omega_{s,k};;f_\nu)|^2\\ &= |\sum_{j=0}^{n-1}d^{(s)}_{\nu,j,\alpha}\alpha^j\omega^j_{s,k}|^2\end{aligned}$$

where we have set

$$d^{(s)}_{\nu,j,\alpha}=b^{(s)}_{\nu,j,\alpha}-p_{\nu,j}c_j,\qquad 0\le j\le n-1. \tag{4.4.9}$$

From (4.2.9) it follows that

$$\begin{aligned}\sum_{k=0}^{n-1}|\mathcal{L}_\nu Q_{n-1}(\alpha\omega_{s,k})-f_\nu(\alpha\omega_{s,k})|^2 &= \sum_{k=0}^{n-1}|\sum_{j=0}^{n-1}d^{(s)}_{\nu,j,\alpha}\alpha^j\omega^j_{s,k}|^2\\ &= n\sum_{j=0}^{n-1}|d^{(s)}_{\nu,j,\alpha}|^2|\alpha|^{2j}.\end{aligned} \tag{4.4.10}$$

If we put

$$\left.\begin{aligned}&c_j=\rho_je^{i\theta_j}, \qquad j=0,1,\ldots,n-1\\ &b^{(s)}_{\nu,j,\alpha}=\sigma_{\nu,j,\alpha}e^{i\phi_{\nu,j,\alpha}},\qquad j=0,1,\ldots,n-1;\ \nu=0,1,\ldots,r-1\end{aligned}\right\} \tag{4.4.11}$$

then from (4.4.9), it follows that

$$|d^{(s)}_{\nu,j,\alpha}|^2=p^2_{\nu,j}\rho^2_j+\sigma^2_{\nu,j,\alpha}-2p_{\nu,j}\rho_j\sigma_{\nu,j,\alpha}cos(\theta_j-\phi_{\nu,j,\alpha}),$$

for $j=0,1,\ldots,n-1$. Thus from (4.10) the problem (4.4.6) reduces to finding the minimum of the following

$$\sum_{\nu=0}^{r-1}\left\{\sum_{j=0}^{n-1}|\alpha|^{2j}p^2_{\nu,j}\rho^2_j+\sum_{j=0}^{n-1}|\alpha|^{2j}\sigma^2_{\nu,j,\alpha}-2\sum_{j=0}^{n-1}|\alpha|^{2j}p_{\nu,j}\rho_j\sigma_{\nu,j,\alpha}cos(\theta_j-\phi_{\nu,j,\alpha})\right\} \tag{4.4.12}$$

where ρ_j runs over the reals and $0 \le \theta_j \le 2\pi$. Differentiating (4.4.12) with respect to ρ_j and θ_j we get the following system of equations to determine ρ_j and θ_j :

$$\left.\begin{array}{l} \rho_j \sum_{\nu=0}^{r-1}(p_{\nu,j})^2 - \sum_{\nu=0}^{r-1} p_{\nu,j}\sigma_{\nu,j,\alpha} cos(\theta_j - \phi_{\nu,j,\alpha}) = 0 \\ \sum_{\nu=0}^{r-1} p_{\nu,j}\sigma_{\nu,j,\alpha} sin(\theta_j - \phi_{\nu,j,\alpha}) = 0, \end{array}\right\} \quad (4.4.13)$$

adding these equations we have

$$\rho_j e^{i\theta_j} \sum_{\nu=0}^{r-1}(p_{\nu,j})^2 - \sum_{\nu=0}^{r-1} p_{\nu,j}\sigma_{\nu,j,\alpha} e^{i\phi_{\nu,j,\alpha}} = 0$$

which gives

$$c_j = c_j^{(s)}(\alpha) = \sum_{\nu=0}^{r-1} p_{\nu,j} b_{\nu,j,\alpha}^{(s)} / (\sum_{\nu=0}^{r-1}(p_{\nu,j})^2),$$

and hence the result.

proof of Lemma 4.4.1 : Here also the proof is on the same lines as that of Lemma 4.2.1. Thus we give only the main steps. From (4.4.4) it follows that (4.4.1) reduces to

$$\sum_{\nu=0}^{r-1}\sum_{k=0}^{n-1} |\sum_{j=0}^{n-1} (j)_\nu c_j^{(s)}(\alpha)(\alpha^j \omega_{s,k}^j) - f_\nu(\alpha^j \omega_{s,k})|^2,$$

where $f_\nu(z) = z^\nu f^{(\nu)}(z), \nu = 0, 1, \ldots, r-1$. Thus from Lemma 4.4.2

$$p_{\nu,j} = (j)_\nu, \qquad \nu = 0, 1, \ldots, r-1, \; j = 0, 1, \ldots, n-1.$$

Since $f \in A_\rho$, we have

$$f_\nu(z) = z^\nu f^{(\nu)}(z) = \frac{\nu!}{2\pi i} \int_\Gamma \frac{f(t) z^\nu}{(t-z)^{\nu+1}} dt$$

where Γ is the circle $|t| = R, 1 < R < \rho$ such that $|\alpha| < R$. Then

$$f_\nu(\alpha\omega_{s,k}) = \frac{\nu!}{2\pi i} \int_\Gamma \frac{f(t)(\alpha\omega_{s,k})^\nu}{(t - (\alpha\omega_{s,k}))^{\nu+1}} dt.$$

Now since

$$\Pi_{k=0}^{n-1}(z - \alpha\omega_{s,k}) = z^n - \alpha^n \Omega_s,$$

where $\Omega_s := (\omega_{s,k})^n$ is a m^{th} root of unity for $k = 0, \ldots, n-1$. Hence if

$$g(\alpha\omega_{s,k}) = \frac{\nu!(\alpha\omega_{s,k})^\nu}{(t - (\alpha\omega_{s,k}))^{\nu+1}},$$

then by Hermite interpolating formula we have

$$\begin{aligned} L'_{n-1,s}(z, \alpha, g) &= \frac{1}{2\pi i} \int_{\Gamma'} \frac{\nu! y^\nu}{(t-y)^{\nu+1}} \frac{y^n - z^n}{(y-z)(y^n - \alpha^n \Omega_s)} dy \\ &= (-1)^\nu \sum_{k=0}^{n-1} \left(\frac{d^\nu}{dt^\nu} \frac{t^{\nu+n-k-1} z^k}{(t^n - \alpha^n \Omega_s)} \right) \end{aligned}$$

where $\Gamma' : |y| = R', 1 < R' < R$ and R' is such that $|\alpha| < R'$ Thus,

$$L'_{n-1,s}(z,\alpha;f_\nu) = \frac{1}{2\pi i}\int_{\Gamma'} f(t)\left\{(-1)^\nu \sum_{j=0}^{n-1} \frac{d^\nu}{dt^\nu}\left(\frac{t^{n+\nu-j-1}}{t^n - \alpha^n \Omega_s}\right) z^j\right\} dt.$$

As seen earlier

$$(-1)^\nu \frac{d^\nu}{dt^\nu}\left(\frac{t^{n+\nu-j-1}}{t^n - \alpha^n\Omega_s}\right) = \sum_{\lambda=0}^{\infty} \alpha^{n\lambda}\Omega_s^\lambda (j+\lambda n)_\nu t^{-\lambda n - j - 1}.$$

Thus

$$\begin{aligned}
L'_{n-1,s}(z;f_\nu) &= \frac{1}{2\pi i}\int_{\Gamma'} f(t)(\sum_{j=0}^{n-1}\sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu \alpha^{n\lambda}\Omega_s^\lambda t^{-\lambda n-j-1} z^j)dt \\
&= \sum_{j=0}^{n-1}\sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu \alpha^{n\lambda}\Omega_s^\lambda z^j \frac{1}{2\pi i}\int_{\Gamma'} f(t) t^{-\lambda n - j - 1} dt \\
&= \sum_{j=0}^{n-1}\sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu \alpha^{n\lambda}\Omega_s^\lambda a_{j+\lambda n} z^j.
\end{aligned}$$

Hence

$$b_{\nu,j,\alpha}^{(s)} = \sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu \alpha^{n\lambda}\Omega_s^\lambda a_{j+\lambda n}.$$

Whence

$$\begin{aligned}
c_j^{(s)} &= \frac{\sum_{\nu=0}^{r-1}(j)_\nu \sum_{\lambda=0}^{\infty}(j+\lambda n)_\nu \alpha^{n\lambda}\Omega_s^\lambda a_{j+\lambda n}}{\sum_{\nu=0}^{r-1}(j)_\nu (j)_\nu} \\
&= \frac{1}{B_{0,j}(r)}\sum_{\lambda=0}^{\infty} B_{\lambda,j}(r) a_{j+\lambda n}\alpha^{n\lambda}\Omega_s^\lambda, \qquad j = 0,1,\ldots,n-1,
\end{aligned}$$

where

$$B_{\lambda,j}(r) = \sum_{i=0}^{r-1}(j)_i (j+\lambda n)_i, \qquad (j)_i = j(j-1),\ldots,(j-i+1).$$

giving the required result.

Thus from (4.4.1)

$$\begin{aligned}
G_{n-1,r}(z,\alpha;f) &= \frac{1}{m}\sum_{s=0}^{m-1} G_{n-1,r}^s(z,\alpha;f) \\
&= \frac{1}{m}\sum_{s=0}^{m-1}\sum_{j=0}^{n-1}\frac{1}{B_{0,j}(r)}\sum_{\lambda=0}^{\infty} B_{\lambda,j}(r) a_{j+\lambda n}\alpha^{n\lambda}\Omega_s^\lambda z^j \\
&= \sum_{j=0}^{n-1}\frac{1}{B_{0,j}(r)}\sum_{\lambda=0}^{\infty} B_{\lambda,j}(r) a_{j+\lambda n}\alpha^{n\lambda} z^j \frac{1}{m}\sum_{s=0}^{m-1}\Omega_s^\lambda.
\end{aligned}$$

Now using (4.2.14) Thus,

$$G_{n-1,r}(z,\alpha;f) = \sum_{j=0}^{n-1}\frac{1}{B_{0,j}(r)}\sum_{\lambda=0}^{\infty} B_{\lambda m,j}(r) a_{j+\lambda mn}\alpha^{n\lambda m} z^j. \tag{4.4.14}$$

4.5 Now let $\alpha, \gamma \in D_\rho$ be two arbitrary points, and let $f \in A_\rho$. For positive integers m and n set

$$\omega_{s,k} = e^{\frac{2\pi i}{mn}(km+s)},$$

for $k = 0, \ldots, n-1$ and $s = 0, \ldots, m-1$. Let $G_{n-1,r}(z, \alpha; f)$ is the polynomial

$$G_{n-1,r}(z, \alpha; f) = \frac{1}{m} \sum_{s=0}^{m-1} G^s_{n-1,r}(z, \alpha; f)$$

where $G^s_{n-1,r}(z, \alpha; f)$ is polynomial which minimizes

$$\sum_{\nu=0}^{r-1} \sum_{k=0}^{n-1} |Q^{(\nu)}_{n-1}(\alpha\omega_{s,k}) - f^{(\nu)}(\alpha\omega_{s,k})|^2,$$

over all polynomials $Q_{n-1} \in \Pi_{n-1}$.

Further, we assume

$$d = ln + p, \qquad l \geq 1, r_1 \leq p/n < 1; \qquad p/n = r_1 + \mathcal{O}(\frac{1}{n}),$$

and where p is integer, and $r_1 \in [0, 1)$ is a given constant.

For b a fixed positive integer let

$$\eta_{q,k} = e^{\frac{2\pi i}{db}(bk+q)}, \;\; q = 0, \ldots, b-1, \;\; k = 0, \ldots, d-1.$$

Let $G_{d-1,r}(z, \gamma; f)$ is the polynomial

$$G_{d-1,r}(z, \alpha; f) = \frac{1}{b} \sum_{q=0}^{b-1} G^q_{d-1,r}(z, \gamma; f)$$

where $G^q_{d-1,r}(z, \gamma; f)$ is polynomial which minimizes

$$\sum_{\nu=0}^{r-1} \sum_{k=0}^{d-1} |Q^{(\nu)}_{d-1}(\gamma\eta_{q,k}) - f^{(\nu)}(\gamma\eta_{q,k})|^2,$$

over all polynomials $Q_{s-1} \in \Pi_{s-1}$.
d d

Let us denote

$$\Theta^{\alpha,\gamma}_{n-1,d,r}(z; f) = G_{n-1,r}(z, \alpha; f) - G_{n-1,r}(z, \alpha; G_{d-1,1}(z, \gamma; f)),$$

$$g_{\alpha,\gamma}(R) = \overline{\lim_{n\to\infty}} \max_{|z|=R} |\Theta^{\alpha,\gamma}_{n-1,d,r}(z; f)|^{1/n}.$$

In this section we give exact estimate of $g_{\alpha,\gamma}(R)$ for $R \geq \rho$ and $R < \rho$. The result generalises Theorem 4.3.1.

Now by the definition β is the smallest positive integer such that $\beta m > l-1$, thus clearly $\beta m \geq l$. Let

$$K_{\alpha,\gamma}(R,\rho) = \begin{cases} max\left(|\frac{\alpha}{\rho}|^{m(\beta+1)}, |\frac{\alpha}{\rho}|^{\beta m}|\frac{R}{\rho}|^{r_1}, |\frac{\gamma}{\rho}|^{(\beta m+r_1)b}\right) & \text{if } 0<|z|<\rho \\ max\left(|\frac{\alpha}{\rho}|^{\beta m}|\frac{R}{\rho}|, |\frac{\gamma}{\rho}|^{(\beta m+r_1)b}|\frac{R}{\rho}|\right) & \text{if } |z|\geq\rho \end{cases}$$

and

$$K'_{\alpha,\gamma}(R,\rho) = \begin{cases} max\left(|\frac{\alpha}{\rho}|^{\beta m}, |\frac{\gamma}{\rho}|^{(l+r_1)b}\right) & \text{if } 0<|z|<\rho \\ max\left(|\frac{\alpha}{\rho}|^{\beta m}|\frac{R}{\rho}|, |\frac{\gamma}{\rho}|^{(l+r_1)b}|\frac{R}{\rho}|\right). & \text{if } |z|\geq\rho \end{cases}$$

Then

Theorem 4.5.1 *For $l=\beta m$, b a fixed poitive integer if $d=d_n=ln+p, p=p_n=r_1n+\mathcal{O}(1), 0\leq r_1<1$, and for each $\alpha,\gamma\in D_\rho$ if $|\alpha/\rho|^{\beta m}\neq|\gamma/\rho|^{(l+r_1)b}$ and for $r_1\neq 0$ if $|\alpha/\rho|^{m(\beta+1)}\neq|\gamma/\rho|^{(l+r_1)b}$ then for each $f\in A_\rho$*

$$g_{\alpha,\gamma}(R)=K_{\alpha,\gamma}(R,\rho), \qquad R>0.$$

and

Theorem 4.5.2 *For $l>\beta m$, b a fixed positive integer if $d=d_n=ln+p, p=p_n=r_1n+\mathcal{O}(1), 0\leq r_1<1$, and for each $\alpha,\gamma\in D_\rho$ if $|\alpha/\rho|^{\beta m}\neq|\gamma/\rho|^{(l+r_1)b}$ then for each $f\in A_\rho$*

$$g_{\alpha,\gamma}(R)=K'_{\alpha,\gamma}(R,\rho), \qquad R>0.$$

Note that for $r=1, m=1, b=1$,

$$G_{n-1,r}(z,\alpha;f)=L_{n-1}(z,\alpha;f)$$

and

$$G_{d-1,1}(z,\gamma;f)=L_{d-1}(z,\gamma;f).$$

Remark 4.5.1 For the special case $r=1, b=1, m=1$ Theorem 4.5.1 reduces to Theorem 4.1.2.

Next, for $\alpha=1, \gamma=0, p=0, b=1$

$$G_{n-1,r}(z,\alpha;f)=G_{n-1,r}(z;f)$$

and

$$G_{d-1,1}(z,\gamma;f) = S_{d-1}(z;f) = S_{ln-1}(z;f)$$

Thus

$$G_{n-1,r}(z, G_{d-1,1}(z,\gamma;f)) = G_{n-1,r}(S_{ln-1}(z;f)).$$

Now from (4.2.2)

$$G_{n-1,r}(z; S_{ln-1}(z;f)) = \frac{1}{m}\sum_{s=0}^{m-1} G^s_{n-1,r}(z; S_{ln-1}(z,;f)),$$

and since

$$S_{ln-1}(z;f) = \sum_{k=0}^{ln-1} a_k z^k = \sum_{k=0}^{n-1}\sum_{j=0}^{l-1} a_{k+nj} z^{k+nj},$$

hence from (4.2.4)

$$G^s_{n-1,r}(z; S_{ln-1}(z;f)) = \sum_{k=0}^{n-1} c_k^{(s)} z^k$$

where

$$c_k^{(s)} = \frac{1}{B_{0,k}(r)}\sum_{\lambda=0}^{l-1} B_{\lambda,k}(r)\Omega_s^\lambda a_{k+\lambda n}, \qquad k = 0, 1, \ldots, n-1.$$

Thus,

$$G_{n-1,r}(z; S_{ln-1}(z;f)) = \frac{1}{m}\sum_{s=0}^{m-1}\sum_{k=0}^{n-1}\frac{1}{B_{0,k}(r)}\sum_{\lambda=0}^{l-1} B_{\lambda,k}(r)\Omega_s^\lambda a_{k+\lambda n} z^k$$

which together with (4.2.14) and the definition of β yeilds

$$G_{n-1,r}(z; S_{ln-1}(z;f)) = \sum_{k=0}^{n-1}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}(r)}{B_{0,k}(r)} a_{k+jmn} z^k.$$

Hence,

Remark 4.5.2 For the special case $\alpha = 1, \gamma = 0, p = 0, b = 1$ Theorem 4.5.1 and Theorem 4.5.2 reduces to Theorem 4.3.1 .

Proof of Theorem 4.5.1 : Here and after we consider $B_{jm,k} = B_{jm,k}(r)$. From (4.4.14) for $n \equiv d$ and $m \equiv b$

$$\begin{aligned}
G_{d-1,1}(z,\gamma;f) &= \sum_{j=0}^{\infty}\sum_{k=0}^{d-1} a_{k+jbd}\gamma^{jbd} z^k \\
&= \sum_{k=0}^{d-1} d_k z^k, \qquad \text{where} \qquad d_k = \sum_{j=0}^{\infty} a_{k+jbd}\gamma^{jbd} \\
&= \sum_{k=0}^{ln+p-1} d_k z^k \\
&= \sum_{j=0}^{l-1}\sum_{k=0}^{n-1} d_{k+jn} z^{k+jn} + \sum_{k=0}^{p-1} d_{k+ln} z^{k+ln}
\end{aligned}$$

hence

$$G_{n-1,r}(z,\alpha;G_{d-1,1}(z,\gamma;f)) = G_{n-1,r}\left(z,\alpha;\left(\sum_{j=0}^{l-1}\sum_{k=0}^{n-1} d_{k+jn}z^{k+jn}+ + \sum_{k=0}^{p-1} d_{k+ln}z^{k+ln}\right)\right). \tag{4.5.1}$$

Note that

$$\begin{aligned} G_{n-1,r}\left(z,\alpha;\left(\sum_{k=0}^{p-1} d_{k+ln}z^{k+ln}\right)\right) &= \frac{1}{m}\sum_{s=0}^{m-1}\sum_{k=0}^{p-1}\frac{B_{l,k}}{B_{0,k}}d_{k+ln}z^k\Omega_s^{ln} \\ &= \begin{cases} \sum_{k=0}^{p-1}\frac{B_{\beta m,k}}{B_{0,k}}d_{k+\beta mn}z^k & \text{if } l=\beta m \\ 0 & \text{otherwise.}\end{cases} \end{aligned} \tag{4.5.2}$$

Hence for $l=\beta m$

$$\begin{aligned} G_{n-1,r}(z,\alpha;G_{d-1,1}(z,\gamma;f)) &= \sum_{j=0}^{\beta-1}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}d_{k+jmn}\alpha^{jmn}z^k + \\ &\quad \sum_{k=0}^{p-1}\frac{B_{\beta m,k}}{B_{0,k}}d_{k+\beta mn}\alpha^{\beta mn}z^k \\ &= \sum_{j=0}^{\beta-1}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}\sum_{i=0}^{\infty}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn}z^k + \\ &\quad +\sum_{k=0}^{p-1}\frac{B_{\beta m,k}}{B_{0,k}}\sum_{i=0}^{\infty}a_{k+ibd+\beta mn}\gamma^{ibd}\alpha^{\beta mn}z^k \end{aligned} \tag{4.5.3}$$

also from (4.4.14)

$$G_{n-1,r}(z,\alpha;f) = \sum_{j=0}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn}z^k$$

this together with (4.5.3) gives

$$\Theta_{n-1,d,r}^{\alpha,\gamma}(z;f) = \sum_{k=0}^{n-1} D_{k,n}z^k \tag{4.5.4}$$

where

$$\begin{aligned} D_{k,n} &= \begin{cases} -\sum_{i=0}^{\infty}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+ibd+\beta mn}\gamma^{ibd}\alpha^{\beta mn} - \sum_{i=0}^{\infty}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn} \\ \quad +\sum_{j=0}^{\infty}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn} \qquad \text{for } 0\le k\le p_n-1 \\ -\sum_{i=0}^{\infty}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn} + \sum_{j=0}^{\infty}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn} \\ \qquad \text{for } p_n\le k\le n-1 \end{cases} \\ &= \begin{cases} \sum_{j=0}^{\infty}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn} - \sum_{i=0}^{\infty}\sum_{j=0}^{\beta}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn} \\ \qquad \text{for } 0\le k\le p_n-1 \\ \sum_{j=0}^{\infty}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn} - \sum_{i=0}^{\infty}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn} \\ \qquad \text{for } p_n\le k\le n-1 \end{cases} \end{aligned}$$

For $0 \leq k \leq p_n - 1$ let $\epsilon > 0$ be too small that

$$(\rho/(\rho-\epsilon))^{r_1} max \left\{ |\frac{\alpha}{\rho-\epsilon}|^{m(\beta+2)}, |\frac{\gamma}{\rho-\epsilon}|^{(l+r_1)b} |\frac{\alpha}{\rho-\epsilon}|^{m}, |\frac{\gamma}{\rho-\epsilon}|^{2(l+r_1)b} \right\} <$$

$$max \left\{ |\frac{\alpha}{\rho}|^{m(\beta+1)}, |\frac{\gamma}{\rho}|^{(l+r_1)b} \right\} = \Lambda_1.$$

Thus,

$$\begin{aligned}
D_{k,n} &= \sum_{j=0}^{\infty} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} \alpha^{jmn} - \sum_{j=0}^{\beta} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} \alpha^{jmn} - \\
&\quad - \sum_{j=0}^{\beta} \frac{B_{jm,k}}{B_{0,k}} a_{k+bd+jmn} \gamma^{bd} \alpha^{jmn} - \sum_{i=2}^{\infty} \sum_{j=0}^{l} \frac{B_{jm,k}}{B_{0,k}} a_{k+ibd+jmn} \gamma^{ibd} \alpha^{jmn} \\
&= \sum_{j=\beta+1}^{\infty} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} \alpha^{jmn} - \frac{B_{0,k}}{B_{0,k}} a_{k+bd} \gamma^{bd} - \\
&\quad \sum_{j=1}^{\beta} \frac{B_{jm,k}}{B_{0,k}} a_{k+bd+jmn} \gamma^{bd} \alpha^{jmn} - \sum_{i=2}^{\infty} \sum_{j=0}^{\beta} \frac{B_{jm,k}}{B_{0,k}} a_{k+ibd+jmn} \gamma^{ibd} \alpha^{jmn} \\
&= \frac{B_{(\beta+1)m,k}}{B_{0,k}} a_{k+(\beta+1)mn} \alpha^{(\beta+1)mn} - a_{k+bd} \gamma^{bd} + \\
&\quad + \mathcal{O}N(n) \left(\frac{|\alpha|^{(\beta+2)mn}}{(\rho-\epsilon)^{(\beta+2)mn+k}} + \frac{|\gamma|^{bd} |\alpha|^{mn}}{(\rho-\epsilon)^{k+bd+mn}} + \frac{|\gamma|^{2bd}}{(\rho-\epsilon)^{k+2bd}} \right) \\
&= \frac{B_{(\beta+1)m,k}}{B_{0,k}} a_{k+(\beta+1)mn} \alpha^{(\beta+1)mn} - a_{k+bd} \gamma^{bd} + \rho^{-k} \mathcal{O}(N(n)(\sigma\Lambda_1)^n) \qquad (4.5.5)
\end{aligned}$$

where $0 < \sigma < 1$ and $N(n)$ is quanitity dependent of n such that $\lim_{n\to\infty}(N(n))^{1/n} = 1$, further $N(n)$ may not be same at each occurence.

Similarly for $p_n \leq k \leq n-1$ let $\epsilon > 0$ be so small that

$$(\rho/(\rho-\epsilon)) max \left\{ |\frac{\alpha}{\rho-\epsilon}|^{m(\beta+1)}, |\frac{\gamma}{\rho-\epsilon}|^{(l+r_1)b} |\frac{\alpha}{\rho-\epsilon}|^{m}, |\frac{\gamma}{\rho-\epsilon}|^{2(l+r_1)b} \right\} <$$

$$max \left\{ |\frac{\alpha}{\rho}|^{\beta m}, |\frac{\gamma}{\rho}|^{(l+r_1)b} \right\} = \Lambda_2.$$

Thus,

$$\begin{aligned}
D_{k,n} &= \sum_{j=0}^{\infty} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} \alpha^{jmn} - \sum_{i=0}^{\infty} \sum_{j=0}^{\beta-1} \frac{B_{jm,k}}{B_{0,k}} a_{k+ibd+jmn} \gamma^{ibd} \alpha^{jmn} \\
&= \sum_{j=0}^{\infty} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} \alpha^{jmn} - \sum_{j=0}^{\beta-1} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} \alpha^{jmn} - \\
&\quad - \sum_{j=0}^{\beta-1} \frac{B_{jm,k}}{B_{0,k}} a_{k+bd+jmn} \gamma^{bd} \alpha^{jmn} - \sum_{i=2}^{\infty} \sum_{j=0}^{\beta-1} \frac{B_{jm,k}}{B_{0,k}} a_{k+ibd+jmn} \gamma^{ibd} \alpha^{jmn} \\
&= \sum_{j=l}^{\infty} \frac{B_{jm,k}}{B_{0,k}} a_{k+jmn} \alpha^{jmn} - \frac{B_{0,k}}{B_{0,k}} a_{k+bd} \gamma^{bd} -
\end{aligned}$$

$$
\begin{aligned}
&\sum_{j=1}^{l-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+bd+jmn}\gamma^{bd}\alpha^{jmn}-\sum_{i=2}^{\infty}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn}\\
= &\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}\alpha^{\beta mn}-a_{k+bd}\gamma^{bd}+\\
&+\mathcal{O}N(n)\left(\frac{|\alpha|^{(\beta+1)mn}}{(\rho-\epsilon)^{(\beta+1)mn+k}}+\frac{|\gamma|^{bd}|\alpha|^{mn}}{(\rho-\epsilon)^{k+bd+mn}}+\frac{|\gamma|^{2bd}}{(\rho-\epsilon)^{k+2bd}}\right)\\
= &\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}\alpha^{\beta mn}-a_{k+bd}\gamma^{bd}+\rho^{-k}\mathcal{O}(N(n)(\sigma\Lambda_2)^n) \qquad (4.5.6)
\end{aligned}
$$

hence

$$
\begin{aligned}
\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f) &= \alpha^{(\beta+1)mn}\sum_{k=0}^{p_n-1}\frac{B_{(\beta+1)m,k}}{B_{0,k}}a_{k+(\beta+1)mn}z^k+\alpha^{\beta mn}\sum_{k=p_n}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k\\
&\quad -\gamma^{bd_n}\sum_{k=0}^{n-1}a_{k+bd_n}z^k+R_n(z),
\end{aligned}
$$

where

$$
\begin{aligned}
R_n(z) &= \mathcal{O}\left(N(n)\sigma^n\Lambda_1^n\sum_{k=0}^{p_n-1}|z/\rho|^k+N(n)\sigma^n\Lambda_2^n\sum_{k=p_n}^{n-1}|z/\rho|^k\right)\\
&= \begin{cases}\mathcal{O}\left(N(n)(\sigma max(\Lambda_1+\Lambda_2|z/\rho|^{r_1}))^n\right) & \text{if } |z|<\rho\\ \mathcal{O}\left(N(n)(\sigma max(\Lambda_1|z/\rho|^{r_1}+\Lambda_2|z/\rho|))^n\right) & \text{if } |z|\ge\rho\end{cases}
\end{aligned}
$$

hence

$$
\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)=\begin{cases}\mathcal{O}N(n)\left(|\frac{\alpha}{(\rho-\epsilon)}|^{(\beta+1)mn}+|\frac{\alpha}{(\rho-\epsilon)}|^{\beta mn}|\frac{z}{(\rho-\epsilon)}|^{p_n}+|\frac{\gamma}{(\rho-\epsilon)}|^{bd_n}\right)+R_n(z)\\ \qquad\qquad \text{if } 0<|z|<\rho\\ \mathcal{O}N(n)\left(|\frac{\alpha}{(\rho-\epsilon)}|^{(\beta+1)mn}|\frac{z}{(\rho-\epsilon)}|^{p_n}+|\frac{\alpha}{(\rho-\epsilon)}|^{\beta mn}|\frac{z}{(\rho-\epsilon)}|^{n}+\right.\\ \left.|\frac{\gamma}{(\rho-\epsilon)}|^{bd_n}|\frac{z}{(\rho-\epsilon)}|^{n}\right)+R_n(z) \qquad \text{if } |z|\ge\rho\end{cases}
$$

on taking n^{th} root which yields

$$
\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)|^{1/n}\le\begin{cases}max\left(|\frac{\alpha}{(\rho-\epsilon)}|^{(\beta+1)m},|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}|\frac{z}{(\rho-\epsilon)}|^{r_1},|\frac{\gamma}{(\rho-\epsilon)}|^{b(l+r_1)}\right)\\ \qquad\qquad \text{if } 0<|z|<\rho\\ max\left(|\frac{\alpha}{(\rho-\epsilon)}|^{(\beta+1)m}|\frac{z}{(\rho-\epsilon)}|^{r_1},|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}|\frac{z}{(\rho-\epsilon)}|,\right.\\ \left.|\frac{\gamma}{(\rho-\epsilon)}|^{b(l+r_1)}|\frac{z}{(\rho-\epsilon)}|\right) \qquad \text{if } |z|\ge\rho.\end{cases}
$$

Since $l=\beta m$, hence

$$
\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)|^{1/n}\le K_{\alpha,\gamma}(R,\rho-\epsilon)
$$

since ϵ is arbitrarily small hence

$$
g_{\alpha,\gamma}(R)\le K_{\alpha,\gamma}(R,\rho).
$$

For the opposite inequality to show that $g_{\alpha,\gamma}(R) \geq K_{\alpha,\gamma}(R,\rho)$.

Now from (4.5.4) with Caushi's formula we have

$$D_{k,n} = \frac{1}{2\pi i}\int_{|z|=R} \frac{\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)}{z^{k+1}}dz$$

and therefore

$$R^k|D_{k,n}| \leq \max_{|z|=R}|\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)|, \qquad 0 \leq k \leq n-1, \qquad R>0. \tag{4.5.7}$$

Now $k+bd_n = k+b(ln+p) = k+lbn+pb$ it is clear that there exists an integer $C>0$ such that for $n-C \leq k \leq n-1$, the sequences $\{k+\beta mn\}$ and $\{k+bd_n\}$ takes all positive integer values. Since $p_n < n-C$ for sufficiently large n and $|\frac{\alpha}{\rho}|^{\beta m} \neq |\frac{\gamma}{\rho}|^{(l+r_1)b}$, hence from (4.5.6)

$$\overline{\lim_{n\to\infty}}\{\max_{n-C\leq k\leq n-1}|D_{k,n}|\}^{1/n} = \frac{1}{(\rho-\epsilon)}max(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b})$$

with (4.5.7) which gives

$$\frac{R}{(\rho-\epsilon)}max(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}) \leq g_{\alpha,\gamma}(R). \tag{4.5.8}$$

Similarly we can choose $C>0$ such that the sequences $\{k+\beta mn\}$ and $\{k+bd_n\}$ assumes all positive integer values for $p_n \leq k \leq p_n+C$ and $p_n+C<n$ for sufficiently large n, hence from (4.5.6),

$$\overline{\lim_{n\to\infty}}\{\max_{p_n\leq k\leq p_n+C}|D_{k,n}|\}^{1/n} = \frac{1}{(\rho-\epsilon)^{r_1}}max(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b})$$

which together with (4.5.7) give

$$|\frac{R}{(\rho-\epsilon)}|^{r_1}max(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}) \leq g_{\alpha,\gamma}(R). \tag{4.5.9}$$

For the case $r_1 = 0$ from (4.5.8) and (4.5.9) we have

$$g_{\alpha,\gamma}(R) \geq K_{\alpha,\gamma}(R,(\rho-\epsilon)).$$

Let now $r_1 > 0$. As $k+bd_n = k+lbn+pb$, choose $C>0$ such that $\{k+bd_n\}$ and $\{k+(\beta+1)mn\}$ for $0 \leq k \leq C$ assume all positive integer values. But for n sufficiently large, we have $C<p_n$, and since $|\frac{\alpha}{\rho}|^{m(\beta+1)} \neq |\frac{\gamma}{\rho}|^{(l+r_1)b}$, thus from (4.5.5) we have

$$\overline{\lim_{n\to\infty}}\{\max_{0\leq k\leq C}|D_{k,n}|\}^{1/n} = max(|\frac{\alpha}{(\rho-\epsilon)}|^{m(\beta+1)}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b})$$

which together with (4.5.7) gives

$$max(|\frac{\alpha}{(\rho-\epsilon)}|^{m(\beta+1)}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}) \leq g_{\alpha,\gamma}(R). \tag{4.5.10}$$

Similarly if $C > 0$ is such that the sequence $\{k + bd_n\}$ and $\{k + (\beta + 1)mn\}$ for $p_n - C \leq k \leq p_n - 1$ assumes all positive integer values, from (4.5.5) we obtain

$$\overline{\lim_{n\to\infty}}\{\max_{p_n-C\leq k\leq p_n-1} |D_{k,n}|\}^{1/n} = \frac{1}{(\rho-\epsilon)^{r_1}} max(|\frac{\alpha}{(\rho-\epsilon)}|^{m(\beta+1)}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}).$$

This together with (4.5.7) gives

$$|\frac{R}{(\rho-\epsilon)}|^{r_1} max(|\frac{\alpha}{(\rho-\epsilon)}|^{m(\beta+1)}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}) \leq g_{\alpha,\gamma}(R). \tag{4.5.11}$$

From (4.5.8),(4.5.9), (4.5.10) and (4.5.11) it follows that for $0 < r_1 < 1$, for the case $0 < R < \rho$ we have

$$max\left\{|\frac{\alpha}{(\rho-\epsilon)}|^{m(\beta+1)}, |\frac{R}{(\rho-\epsilon)}|^{r_1}|\frac{\alpha}{(\rho-\epsilon)}|^{\beta l}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}\right\} \leq g_{\alpha,\gamma}(R)$$

and for $R \geq \rho$ we have

$$|\frac{R}{(\rho-\epsilon)}| max\left\{|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}\right\} \leq g_{\alpha,\gamma}(R).$$

Since ϵ is arbitrary small and $l = \beta m$ we have

$$K_{\alpha,\gamma}(R,\rho) \leq g_{\alpha,\gamma}(R)$$

which completes the proof.

Proof of Theorem 4.5.2 : From (4.5.1) we have

$$G_{n-1,r}(z,\alpha;G_{d-1,1}(z,\gamma;f)) = G_{n-1,r}\left(z,\alpha;\left(\sum_{j=0}^{l-1}\sum_{k=0}^{n-1} d_{k+jn}z^{k+jn} + +\sum_{k=0}^{p-1} d_{k+ln}z^{k+ln}\right)\right).$$

From hypothesis $l > \beta m$ that is $l \neq \beta m$, hence from (4.5.2)

$$G_{n-1,r}\left(z,\alpha;\left(\sum_{k=0}^{p-1} d_{k+ln}z^{k+ln}\right)\right) = 0.$$

Thus, for $l > \beta m$

$$\begin{aligned} G_{n-1,r}(z,\alpha;G_{d-1,1}(z,\gamma;f)) &= \sum_{j=0}^{\beta-1}\sum_{k=0}^{n-1} \frac{B_{jm,k}}{B_{0,k}} d_{k+jmn}\alpha^{jmn}z^k \\ &= \sum_{j=0}^{\beta-1}\sum_{k=0}^{n-1} \frac{B_{jm,k}}{B_{0,k}} \sum_{i=0}^{\infty} a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn}z^k \end{aligned} \tag{4.5.12}$$

also from (4.4.14)

$$G_{n-1,r}(z,\alpha;f)=\sum_{j=0}^{\infty}\sum_{k=0}^{n-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn}z^k$$

this together with (4.5.12) gives

$$\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)=\sum_{k=0}^{n-1}D_{k,n}z^k \tag{4.5.13}$$

where

$$D_{k,n}=\sum_{j=0}^{\infty}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn}-\sum_{i=0}^{\infty}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn}$$

for $0\le k\le n-1$. For $0\le k\le n-1$ let $\epsilon>0$ be so small that

$$(\rho/(\rho-\epsilon))max\left\{|\frac{\alpha}{\rho-\epsilon}|^{m(\beta+1)},|\frac{\gamma}{\rho-\epsilon}|^{(l+r_1)b}|\frac{\alpha}{\rho-\epsilon}|^{m},|\frac{\gamma}{\rho-\epsilon}|^{2(l+r_1)b}\right\}<$$

$$max\left\{|\frac{\alpha}{\rho}|^{\beta m},|\frac{\gamma}{\rho}|^{(l+r_1)b}\right\}=\Lambda.$$

Thus,

$$\begin{aligned}
D_{k,n} &= \sum_{j=0}^{\infty}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn}-\sum_{i=0}^{\infty}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn}\\
&= \sum_{j=0}^{\infty}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn}-\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn}-\\
&\quad -\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+bd+jmn}\gamma^{bd}\alpha^{jmn}-\sum_{i=2}^{\infty}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn}\\
&= \sum_{j=l}^{\infty}\frac{B_{jm,k}}{B_{0,k}}a_{k+jmn}\alpha^{jmn}-\frac{B_{0,k}}{B_{0,k}}a_{k+bd}\gamma^{bd}-\\
&\quad \sum_{j=1}^{l-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+bd+jmn}\gamma^{bd}\alpha^{jmn}-\sum_{i=2}^{\infty}\sum_{j=0}^{\beta-1}\frac{B_{jm,k}}{B_{0,k}}a_{k+ibd+jmn}\gamma^{ibd}\alpha^{jmn}\\
&= \frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}\alpha^{\beta mn}-a_{k+bd}\gamma^{bd}+\\
&\quad +\mathcal{O}N(n)\left(\frac{|\alpha|^{(\beta+1)mn}}{(\rho-\epsilon)^{(\beta+1)mn+k}}+\frac{|\gamma|^{bd}|\alpha|^{mn}}{(\rho-\epsilon)^{k+bd+mn}}+\frac{|\gamma|^{2bd}}{(\rho-\epsilon)^{k+2bd}}\right)\\
&= \frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}\alpha^{\beta mn}-a_{k+bd}\gamma^{bd}+\rho^{-k}\mathcal{O}(N(n)(\sigma\Lambda)^n)
\end{aligned} \tag{4.5.14}$$

hence

$$\begin{aligned}
\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f) &= \alpha^{\beta mn}\sum_{k=0}^{n-1}\frac{B_{\beta m,k}}{B_{0,k}}a_{k+\beta mn}z^k\\
&\quad -\gamma^{bd_n}\sum_{k=0}^{n-1}a_{k+bd_n}z^k+R_n(z)
\end{aligned}$$

where

$$\begin{aligned}
R_n(z) &= \mathcal{O}\left(N(n)\sigma^n\Lambda^n\sum_{k=0}^{n-1}|z/\rho|^k\right)\\
&= \begin{cases} \mathcal{O}\left(N(n)(\sigma\Lambda)^n\right) & \text{if } |z|<\rho \\ \mathcal{O}\left(N(n)(\sigma\Lambda|z/\rho|)^n\right). & \text{if } |z|\geq\rho \end{cases}
\end{aligned}$$

Hence

$$\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)=\begin{cases} \mathcal{O}N(n)\left(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta mn}+|\frac{\gamma}{(\rho-\epsilon)}|^{bd_n}\right)+R_n(z) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0<|z|<\rho \\ \mathcal{O}N(n)\left(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta mn}|\frac{z}{(\rho-\epsilon)}|^n+|\frac{\gamma}{(\rho-\epsilon)}|^{bd_n}|\frac{z}{(\rho-\epsilon)}|^n\right)+R_n(z) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } |z|\geq\rho \end{cases}$$

on taking n^{th} root which yields

$$\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)|^{1/n}\leq\begin{cases} max\left(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m},|\frac{\gamma}{(\rho-\epsilon)}|^{b(l+r_1)}\right) \\ \qquad\qquad\qquad\qquad\qquad \text{if } 0<|z|<\rho \\ max\left(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}|\frac{z}{(\rho-\epsilon)}|,|\frac{\gamma}{(\rho-\epsilon)}|^{b(l+r_1)}|\frac{z}{(\rho-\epsilon)}|\right) \\ \qquad\qquad\qquad\qquad\qquad \text{if } |z|\geq\rho. \end{cases}$$

Hence,

$$\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)|^{1/n}\leq K'_{\alpha,\gamma}(R,\rho-\epsilon)$$

since ϵ is arbitrarily small hence

$$g_{\alpha,\gamma}(R)\leq K'_{\alpha,\gamma}(R,\rho).$$

For the opposite inequality to show that $g_{\alpha,\gamma}(R)\geq K'_{\alpha,\gamma}(R,\rho)$.

Now from (4.5.13) with Caushi's formula we have

$$D_{k,n}=\frac{1}{2\pi i}\int_{|z|=R}\frac{\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)}{z^{k+1}}dz$$

and therefore

$$R^k|D_{k,n}|\leq\max_{|z|=R}|\Theta^{\alpha,\gamma}_{n-1,d,r}(z;f)|,\qquad 0\leq k\leq n-1,\qquad R>0. \tag{4.5.15}$$

Now $k+bd_n=k+b(ln+p)=k+lbn+pb$ it is clear that there exists an integer $C>0$ such that for $n-C\leq k\leq n-1$, the sequences $\{k+\beta mn\}$ and $\{k+bd_n\}$ takes all positive

integer values. Since $p_n < n - C$ for sufficiently large n and $|\frac{\alpha}{\rho}|^{\beta m} \neq |\frac{\gamma}{\rho}|^{(l+r_1)b}$, hence from (4.5.14)

$$\overline{\lim_{n\to\infty}}\{\max_{n-C\leq k\leq n-1} |D_{k,n}|\}^{1/n} = \frac{1}{(\rho-\epsilon)} max(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b})$$

with (4.5.15) which gives

$$\frac{R}{(\rho-\epsilon)} max(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}) \leq g_{\alpha,\gamma}(R). \tag{4.5.16}$$

Similarly we can choose $C > 0$ such that the sequences $\{k + \beta m n\}$ and $\{k + bd_n\}$ assumes all positidve integer values for $0 \leq k \leq C$ and $C < n$ for sufficiently large n, hence from (4.5.14),

$$\overline{\lim_{n\to\infty}}\{\max_{0\leq k\leq C} |D_{k,n}|\}^{1/n} = max(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}).$$

Thus from (4.5.15)

$$max(|\frac{\alpha}{(\rho-\epsilon)}|^{\beta m}, |\frac{\gamma}{(\rho-\epsilon)}|^{(l+r_1)b}) \leq g_{\alpha,\gamma}(R) \tag{4.5.17}$$

which together with (4.5.16) give

$$g_{\alpha,\gamma}(R) \geq K'_{\alpha,\gamma}(R, (\rho-\epsilon)).$$

Since ϵ is arbitrary small we have

$$K'_{\alpha,\gamma}(R, \rho) \leq g_{\alpha,\gamma}(R)$$

which completes the proof.

Chapter 5

WALSH OVERCONVERGENCE USING DERIVATIVES OF HERMITE INTERPOLATING POLYNOMIALS

5.1 Let $\rho > 1$, denote by A_ρ and R_ρ the set of all functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with the coefficients satisfying

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \rho^{-1} \qquad \text{and} \qquad \overline{\lim_{n\to\infty}} |a_n|^{1/n} \leq \rho^{-1}$$

respectively. In this chapter we consider Hermite interpolation. For a fixed integer $r \geq 1$ and for every $n \geq 1$, let $h_{rn-1}(z; f) \in \Pi_{rn-1}$ denote the Hermite interpolant to f in the n^{th} roots of unity. That is

$$h^{\nu}_{rn-1}(\omega_k; f) = f^{\nu}(\omega_k), \qquad \nu = 0, \ldots, r-1, \ k = 0, \ldots, n-1 \tag{5.1.1}$$

where $\omega_k^n = 1$. Then from [12]

$$h_{rn-1}(z; f) = \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{\infty} \beta_{j,r}(z^n) \sum_{k=0}^{n-1} a_{k+(r+j-1)n} z^k, \tag{5.1.2}$$

where

$$\beta_{j,r}(z) = \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z-1)^k, j \geq 1. \tag{5.1.3}$$

If we set

$$H_{rn-1,0}(z; f) = \sum_{k=0}^{rn-1} a_k z^k \tag{5.1.4}$$

and for each $j \geq 1$ set

$$H_{rn-1,j}(z;f) = \beta_{j,r}(z^n) \sum_{k=0}^{n-1} a_{k+(r+j-1)n} z^k. \tag{5.1.5}$$

Next for $l \geq 1$ denote

$$\Delta_{rn-1,l}(z;f) = h_{rn-1}(z;f) - \sum_{j=0}^{l-1} H_{rn-1,j}. \tag{5.1.6}$$

Then from (5.1.2), (5.1.4) and (5.1.5), (5.1.6) can be written as

$$\Delta_{rn-1,l}(z;f) = \sum_{j=l}^{\infty} \beta_{j,r}(z^n) \sum_{k=0}^{n-1} a_{k+(r+j-1)n} z^k. \tag{5.1.7}$$

Let

$$\begin{aligned} K^1_{l,r}(|z|,\rho) &= \rho^{-1-(l-1)/r} max(1, |z|^{1-1/r}, |z|\rho^{-1/r}) \\ &= \begin{cases} 1/\rho^{1+(l-1)/r} & |z| \leq 1, \\ |z|^{1-1/r}/\rho^{1+(l-1)/r} & 1 \leq |z| \leq \rho, \\ |z|/\rho^{1+l/r} & \rho \leq |z|. \end{cases} \end{aligned} \tag{5.1.8}$$

In the Lagrange case ($r = 1$), we have only two domains in (5.1.8). When $r > 1$, in the Hermite case we have always three domains in (5.1.8).

Set

$$D_{l,r}(R;f) = \overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta_{rn-1,l}(z;f)|^{1/rn}. \tag{5.1.9}$$

With these notations Ivanov and Sharma [20] proved

Theorem 5.1.1 *Let $r, l \geq 1$ and let $\rho > 1$. For any $f \in A_\rho$, we have*

$$D_{l,r}(R;f) = K^1_{r,l}(R,\rho), \qquad R > 0. \tag{5.1.10}$$

If we set

$$G_{l,r}(z;f) = \overline{\lim_{n\to\infty}} |\Delta_{rn-1,l}(z;f)|^{1/rn}, \tag{5.1.11}$$

then from (5.1.9), (5.1.10) and (5.1.11)

$$G_{l,r}(z,f) \leq K^1_{l,r}(|z|,\rho).$$

A set Z is an (r,l,ρ)- distinguished set if there exists an $f \in A_\rho$ such that $G_{l,r}(z,f) < K^1_{l,r}(|z|,\rho)$ for every $z \in Z$. Following the idea of [19] introduce the matrices X, Y and

M(X,Y) corresponding to a given set $Z = \{z_j\}_{j=1}^{s}$ in which

$$\begin{cases} |z_j| < \rho, & j = 1, \ldots, \mu; \\ |z_j| > \rho, & j = \mu, \ldots, s; \\ |z_j| \neq 1, & j = 1, \ldots, s, \text{when} \beta_{l,r}(z) \text{ has a zero on} \\ & \qquad\qquad \text{the unit circle.} \end{cases} \tag{5.1.12}$$

$$X = \begin{pmatrix} 1 & z_1 & \ldots & z_1^{r+l-2} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & z_\mu & \ldots & z_\mu^{r+l-2} \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & z_{\mu+1} & \ldots & z_{\mu+1}^{r+l-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & z_s & \ldots & z_s^{r+l-1} \end{pmatrix}$$

The matrices X and Y are of order $(\mu \times r + l - 1$ and $(s - \mu) \times r + l$ respectively. Define

$$M = M(X,Y) = \begin{pmatrix} X & & & \\ & X & & 0 \\ & & \ddots & \\ & 0 & & X \\ Y & & & \\ & Y & & 0 \\ & & \ddots & \\ & 0 & & Y \end{pmatrix},$$

where X occurs $r + l$ times and Y occurs $r + l - 1$ times beginning under the last X. The matrix M is of order $(s(r + l - 1) + \mu) \times (r + l - 1)(r + l)$. Ivanov and Sharma [20] proved

Theorem 5.1.2 [20] *Let the set $Z = \{z_j\}_{j=1}^{s}$ satisfy (5.1.12). Then Z is (r, l, ρ) distinguished iff*

$$rank M < (r + l - 1)(r + l).$$

In this chapter we study $\overline{\lim}_{n\to\infty} \max_{|z|=R} |\Delta^{(t)}_{rn-1,l}(z; f)|^{1/rn}$, where $\Delta^{(t)}_{rn-1}(z; f)$ is the t^{th} derivative of $\Delta_{rn-1,l}(z; f)$. In section 5.2 we give some exact results for $\Delta^{(t)}_{rn-1,l}(z; f)$. Next we introduce the concept of distinguished point of degree t and investigate some relations between the order of pointwise convergence of $\Delta^{(t)}_{rn-1,l}(z; f)$ and the properties of $f(z)$. In section 5.4 we generalize Theorem 5.1.1 for the case that the points of $\{z_j\}_1^s$ can be coincided with each other.

5.2 In this section we give an exact result for $\Delta^{(t)}_{rn-1,l}(z;f)$, which as a particular case give Theorem 5.1.1.

Theorem 5.2.1 *For each $f \in R_\rho(\rho > 1)$, any integers $l \geq 1$ and $t \geq 0$, and any $R > 0$, there holds*

$$\overline{\lim_{n\to\infty}} \max_{|z|=R} |\Delta^{(t)}_{rn-1,l}(z;f)|^{1/rn} \leq K^1_{l,r}(R,\rho), \tag{5.2.1}$$

where $K^1_{l,r}(R,\rho)$ is given by (5.1.8). Equality holds in (5.2.1) iff $f \in A_\rho$.

Proof : Note that from (5.1.3)

$$\begin{aligned}
\beta_{j,r}(z^n) &= \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z^n-1)^k \\
&= \sum_{k=0}^{r-1} \binom{r+j-1}{k} \sum_{\lambda=0}^{k} \binom{k}{\lambda} (-1)^{k-\lambda} z^{n\lambda} \\
&= \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda}, \qquad (5.2.2)
\end{aligned}$$

where

$$C_{\lambda,r}(j) = \sum_{k=\lambda}^{r-1} (-1)^{k-\lambda} \binom{r+j-1}{k} (\binom{k}{\lambda}), \qquad \lambda = 0,\ldots,r-1. \tag{5.2.3}$$

Now set $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then for every z on $|z| = R(R>0)$ from (5.1.7) and (5.2.2) we have

$$\begin{aligned}
\Delta^{(t)}_{rn-1,l}(z;f) &= \left(\sum_{k=0}^{n-1} \sum_{j=l}^{\infty} \beta_{j,r}(z^n) a_{k+(r+j-1)n} z^k \right)^{(t)} \\
&= \left(\sum_{k=0}^{n-1} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{k+(r+j-1)n} z^k \right)^{(t)} \\
&= \sum_{k=t}^{n-1} \sum_{j=l}^{\infty} C_{0,r}(j) a_{k+(r+j-1)n} (k)_t z^{k-t} \\
&\quad + \sum_{k=0}^{n-1} \sum_{j=l}^{\infty} \sum_{\lambda=1}^{r-1} C_{\lambda,r}(j) (k+n\lambda)_t a_{k+(r+j-1)n} z^{k+n\lambda-t} \qquad (5.2.4) \\
&= \mathcal{O} \begin{cases} n^t \frac{1}{(\rho-\epsilon)^{(l+r-1)n}} & \text{if} \quad 1 \leq R \\ n^t \frac{R^{n(r-1)}}{(\rho-\epsilon)^{(l+r-1)n}} & \text{if} \quad 1 < R < \rho \\ n^t \frac{R^{n+n(r-1)}}{(\rho-\epsilon)^{n+(l+r-1)n}} & \text{if} \quad R \geq \rho \end{cases}
\end{aligned}$$

where $(k)_t = k(k-1)\ldots(k-t+1)$, $(k)_0 := 1$. Here and elsewhere ϵ will denote sufficintly

small positive number which may differ at different times. Hence

$$\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Delta^{(t)}_{rn-1,l}(z;f)|^{1/rn} \le \begin{cases} \frac{1}{(\rho-\epsilon)^{1+(l-1)/r}} & \text{if} \quad R\le 1 \\ \frac{R^{1-1/r}}{(\rho-\epsilon)^{1+(l-1)/r}} & \text{if} \quad 1<R<\rho \\ \frac{R}{(\rho-\epsilon)^{1+l/r}} & \text{if} \quad R\ge\rho \end{cases}$$

since ϵ is arbitrary small, we obtain (5.2.1).

To prove the second part we show that equality does not hold in (5.2.1) iff $f\in R_\rho\backslash A_\rho$. First suppose equality does not hold in (5.2.1), then there is some $t'\ge 0$ and $f\in R_\rho$ for which strict inequality holds in (5.2.1).That is

$$\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Delta^{(t')}_{rn-1,l}(z;f)|^{1/rn} < K^1_{l,r}(R,\rho). \tag{5.2.5}$$

Thus from (5.2.4)

$$\begin{aligned} \Delta^{(t')}_{rn-1,l}(z;f) &= \sum_{k=t'}^{n-1}\sum_{j=l}^{\infty} C_{0,r}(j)a_{k+(r+j-1)n}(k)_t z^{k-t'} \\ &+\sum_{k=0}^{n-1}\sum_{j=l}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_{t'}a_{k+(r+j-1)n}z^{k+n\lambda-t'} \\ &= \sum_{k=t'}^{n-1} C_{0,r}(l)a_{k+(r+l-1)n}(k)_t z^{k-t'} \\ &+\sum_{k=0}^{n-1}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(l)(k+n\lambda)_{t'}a_{k+(r+l-1)n}z^{k+n\lambda-t'} \\ &+\sum_{k=t'}^{n-1}\sum_{j=l+1}^{\infty} C_{0,r}(j)a_{k+(r+j-1)n}(k)_t z^{k-t'} \\ &+\sum_{k=0}^{n-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_{t'}a_{k+(r+j-1)n}z^{k+n\lambda-t'}. \end{aligned} \tag{5.2.6}$$

Let $R\ge\rho$ then

$$\begin{aligned} &\sum_{k=n-(r+l)}^{n-1} C_{r-1,r}(l)(k+n(r-1))_{t'}a_{k+(r+l-1)n}z^{k+n(r-1)-t'} \\ &= \Delta^{(t')}_{rn-1,l}(z;f)-\sum_{k=t'}^{n-1} C_{0,r}(l)a_{k+(r+l-1)n}(k)_t z^{k-t'}+ \\ &+\sum_{k=0}^{n-1}\sum_{\lambda=1}^{r-2} C_{\lambda,r}(l)(k+n\lambda)_{t'}a_{k+(r+l-1)n}z^{k+n\lambda-t'} \\ &+\sum_{k=0}^{n-(r+l)-1} C_{r-1,r}(l)(k+n(r-1))_{t'}a_{k+(r+l-1)n}z^{k+n(r-1)-t'}+ \end{aligned}$$

$$+\sum_{k=t'}^{n-1}\sum_{j=l+1}^{\infty} C_{0,r}(j)a_{k+(r+j-1)n}(k)_t z^{k-t'}$$

$$+\sum_{k=0}^{n-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_{t'}a_{k+(r+j-1)n}z^{k+n\lambda-t'}.$$

By using the fact

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{z^{k'}}{z^{k+1}}=\begin{cases}1 & k=k'\\ 0 & k\neq k'\end{cases} \tag{5.2.7}$$

and Cauchy integral formula, we have for $n-(r+l)\leq k\leq n-1$

$$\begin{aligned}
&C_{r-1,r}(l)(k+n(r-1))_{t'}a_{k+(r+l-1)n}\\
&=\tfrac{1}{2\pi i}\int_{|z|=R}\tfrac{\Delta^{(t')}_{rn-1,l}(z;f)}{z^{k+n(r+l-1)-t'+1}}dz-0\\
&+0+0-\tfrac{1}{2\pi i}\int_{|z|=R}\tfrac{\sum_{k'=t'}^{n-1}\sum_{j=l+1}^{\infty}C_{0,r}(j)a_{k'+(r+j-1)n}(k')_t z^{k'-t'}}{z^{k+n(r-1)-t'+1}}dz\\
&+\tfrac{1}{2\pi i}\int_{|z|=R}\tfrac{\sum_{k'=0}^{n-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1}C_{\lambda,r}(j)(k'+n\lambda)_{t'}a_{k'+(r+j-1)n}z^{k'+n\lambda-t'}}{z^{k+n(r-1)-t'+1}}dz\\
&\leq \max_{|z|=R}|\Delta^{(t')}_{rn-1,l}(z;f)|R^{-k-n(r-1)+t'}+\mathcal{O}\left(n^{t'}R^{-k-n(r-1)+t'}\tfrac{R^{n+n(r-1)}}{(\rho-\epsilon)^{n+n(r+l+1-1)}}\right).
\end{aligned}$$

Hence from (5.2.5)

$$\begin{aligned}
&\overline{\lim}_{n\to\infty}|a_{k+(r+l-1)n}|^{k+(r+l-1)n}\\
&< max\left\{\left(R^{-1/r-(1-1/r)}\tfrac{R}{\rho^{1+l/r}}\right)^{\frac{1}{1+l/r}},\left(R^{-1/r-(1-1/r)}\tfrac{R}{(\rho-\epsilon)^{1+(l+1)/r}}\right)^{\frac{1}{1+l/r}}\right\}.
\end{aligned}$$

By choosing $\epsilon>0$ sufficiently small so that

$$(\rho-\epsilon)^{-(1+(l+1)/r)}<\rho^{-(1+l/r)}$$

we have

$$\overline{\lim_{n\to\infty}}|a_{k+(l+r-1)n}|^{\frac{1}{k+(l+r-1)n}}<\frac{1}{\rho},\qquad n-(l+r)\leq k\leq n-1$$

or,

$$\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}}<\frac{1}{\rho}.$$

Thus we have $f\in R_\rho\backslash A_\rho$.

Similarly for $1<R<\rho$ from (5.2.6) we have

$$\sum_{k=0}^{r+l-2}C_{r-1,r}(l)(k+n(r-1))_{t'}a_{k+(r+l-1)n}z^{k+n(r-1)-t'}$$

$$=\Delta^{(t')}_{rn-1,l}(z;f)-\sum_{k=t'}^{n-1}C_{0,r}(l)a_{k+(r+l-1)n}(k)_t z^{k-t'}$$

$$+\sum_{k=0}^{n-1}\sum_{\lambda=1}^{r-2}C_{\lambda,r}(l)(k+n\lambda)_{t'}a_{k+(r+l-1)n}z^{k+n\lambda-t'}$$

$$+\sum_{k=r+l-1}^{n-1} C_{r-1,r}(l)(k+n(r-1))_{t'}a_{k+(r+l-1)n}z^{k+n(r-1)-t'}$$

$$+\sum_{k=t'}^{n-1}\sum_{j=l+1}^{\infty} C_{0,r}(j)a_{k+(r+j-1)n}(k)_t z^{k-t'}$$

$$+\sum_{k=0}^{n-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_{t'}a_{k+(r+j-1)n}z^{k+n\lambda-t'}.$$

By using Cauchy integral formula and (5.2.7), we have for $0 \le k \le r+l-2$

$$\begin{aligned}
&C_{r-1,r}(l)(k+n(r-1))_{t'}a_{k+(r+l-1)n} \\
&= \tfrac{1}{2\pi i}\int_{|z|=R} \tfrac{\Delta^{(t')}_{rn-1,l}(z;f)}{z^{k+n(r-1)-t'+1}}dz - 0 \\
&+0+0-\tfrac{1}{2\pi i}\int_{|z|=R} \tfrac{\sum_{k'=t'}^{n-1}\sum_{j=l+1}^{\infty} C_{0,r}(j)a_{k'+(r+j-1)n}(k')_t z^{k'-t'}}{z^{k+n(r-1)-t'+1}}dz \\
&+\tfrac{1}{2\pi i}\int_{|z|=R} \tfrac{\sum_{k'=0}^{n-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k'+n\lambda)_{t'}a_{k'+(r+j-1)n}z^{k'+n\lambda-t'}}{z^{k+n(r-1)-t'+1}}dz \\
&\le \max_{|z|=R}|\Delta^{(t')}_{rn-1,l}(z;f)|R^{-k-n(r-1)+t'} + \mathcal{O}\left(n^{t'}R^{-k-n(r-1)+t'}\tfrac{R^{n(r-1)}}{(\rho-\epsilon)^{n(r+l+1-1)}}\right).
\end{aligned}$$

Hence from (5.2.5)

$$\begin{aligned}
&\overline{\lim}_{n\to\infty}|a_{k+(r+l-1)n}|^{k+(r+l-1)n} \\
&< max\left\{\left(R^{-(1-1/r)}\tfrac{R^{1-1/r}}{\rho^{1+(l-1)/r}}\right)^{\frac{1}{1+(l-1)/r}}, \left(R^{-(1-1/r)}\tfrac{R^{(1-1/r)}}{(\rho-\epsilon)^{1+l/r}}\right)^{\frac{1}{1+(l-1)/r}}\right\}.
\end{aligned}$$

By choosing $\epsilon > 0$ sufficiently small so that

$$(\rho-\epsilon)^{-(1+l/r)} < \rho^{-(1+(l-1)/r)}$$

we have

$$\overline{\lim_{n\to\infty}}|a_{k+(l+r-1)n}|^{\frac{1}{k+(l+r-1)n}} < \frac{1}{\rho}, \qquad 0 \le k \le r+l-2$$

or,

$$\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}} < \frac{1}{\rho}.$$

Thus we have $f \in R_\rho \backslash A_\rho$.

Further, for $R < 1 < \rho$ from (5.2.6) we have

$$\sum_{k=t'}^{t'+r+l-2} C_{0,r}(l)(k)_{t'}a_{k+(r+l-1)n}z^{k-t'}$$

$$= \Delta^{(t')}_{rn-1,l}(z;f) - \sum_{k=t'+r+l-1}^{n-1} C_{0,r}(l)a_{k+(r+l-1)n}(k)_t z^{k-t'}$$

$$+\sum_{k=0}^{n-1}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(l)(k+n\lambda)_{t'}a_{k+(r+l-1)n}z^{k+n\lambda-t'}$$

$$+\sum_{k=t'}^{n-1}\sum_{j=l+1}^{\infty} C_{0,r}(j)a_{k+(r+j-1)n}(k)_t z^{k-t'}$$

$$+\sum_{k=0}^{n-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_{t'}a_{k+(r+j-1)n}z^{k+n\lambda-t'}.$$

By using Cauchy integral formula and (5.2.7), we have for $t' \leq k \leq t'+r+l-2$

$$\begin{aligned}
&C_{0,r}(l)(k)_{t'}a_{k+(r+l-1)n} \\
&= \tfrac{1}{2\pi i}\int_{|z|=R}\frac{\Delta^{(t')}_{rn-1,l}(z;f)}{z^{k-t'+1}}dz - 0 \\
&+0 - \tfrac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{k'=t'}^{n-1}\sum_{j=l+1}^{\infty} C_{0,r}(j)a_{k'+(r+j-1)n}(k')_t z^{k'-t'}}{z^{k-t+1'}}dz \\
&+\tfrac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{k'=0}^{n-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k'+n\lambda)_{t'}a_{k'+(r+j-1)n}z^{k'+n\lambda-t'}}{z^{k-t'+1}}dz \\
&\leq \max_{|z|=R}|\Delta^{(t')}_{rn-1,l}(z;f)|R^{-k+t'} + \mathcal{O}\left(n^{t'}R^{-k+t'}\tfrac{1}{(\rho-\epsilon)^{n(r+l+1-1)}}\right).
\end{aligned}$$

Hence from (5.2.5)

$$\overline{\lim_{n\to\infty}}|a_{k+(r+l-1)n}|^{k+(r+l-1)n} < max\left\{\left(\frac{1}{\rho^{1+(l-1)/r}}\right)^{\frac{1}{1+(l-1)/r}}, \left(\frac{R^1}{(\rho-\epsilon)^{1+l/r}}\right)^{\frac{1}{1-(l-1)/r}}\right\}.$$

By choosing $\epsilon > 0$ sufficiently small so that

$$(\rho-\epsilon)^{-(1+l/r)} < \rho^{-(1+(l-1)/r)}$$

we have

$$\overline{\lim_{n\to\infty}}|a_{k+(l+r-1)n}|^{\frac{1}{k+(l+r-1)n}} < \frac{1}{\rho}, \qquad t' \leq k \leq t'+r+l-2$$

or,

$$\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}} < \frac{1}{\rho}.$$

Thus we have $f \in R_\rho \backslash A_\rho$.

Next, let $f \in R_\rho \backslash A_\rho$. Thus $f \in R_{\rho_1}$ for some $\rho_1 > \rho$, hence by the first part of Theorem 5.2.1 we have

$$\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Delta^{(t)}_{rn-1,l}(z;f)|^{1/rn} \leq K^1_{l,r}(R,\rho_1).$$

Since $\rho_1 > \rho$ hence by definition

$$K^1_{l,r}(R,\rho_1) < K^1_{l,r}(R,\rho),$$

with above equation which gives

$$\overline{\lim_{n\to\infty}}\max_{|z|=R}|\Delta^{(t)}_{rn-1,l}(z;f)|^{1/rn} < K^1_{l,r}(R,\rho).$$

Thus equality does not hold in (5.2.1) if $f \in R_\rho \backslash A_\rho$.

Corollary 5.2.1 *For each $f \in R_\rho(\rho > 1)$, and any integer $l \geq 1, t \geq 0$, there holds*

$$\lim_{n\to\infty} \Delta^{(t)}_{rn-1,l}(z; f) = 0, \qquad \forall\ |z| < \rho^{1+l/r}.$$

Moreover the result is best possible if $f \in A_\rho$.

Remark 5.2.1 For $t = 0$ Theorem 5.2.1 reduces to Theorem 5.1.1.

Remark 5.2.2 For $r = 1$ Theorem 5.2.1 reduces to Theorem 2.1.6.

5.3 For any integer $t \geq 0$, we set

$$H^t_{l,r}(z; f) := \overline{\lim_{n\to\infty}} |\Delta^{(t)}_{rn-1,l}(z; f)|^{1/rn}.$$

We say that η is an (l, r, ρ)-distinguished point of $f \in A_\rho$ of degree t if

$$H^\nu_{l,r}(\eta; f) < K^1_{l,r}(|\eta|, \rho), \qquad \forall\ \nu = 0, 1, \ldots, t-1,$$

and consider it as t points coincided at η.

Hereafter let $\{\eta_\nu\}^s_{\nu=1}$ be a set of s points in the complex plane and p_ν denote the number of appearence of η_ν in $\{\eta_j\}^\nu_{j=1}$. We prove

Theorem 5.3.1 *If $f \in R_\rho(\rho > 1), l$ is any positive integer, and there are $l + r$ points $\{\eta_\nu\}^{l+r}_{\nu=1}$ in $|z| > \rho$ (or, $l + r - 1$ points $\{\eta_\nu\}^{l+r-1}_{\nu=1}$ in $|z| < \rho$) for which*

$$H^{p_\nu - 1}_{l,r}(\eta_\nu; f) < K^1_{l,r}(|\eta_\nu|, \rho), \qquad \nu = 1, \ldots, l + r (or\ l + r - 1),$$

then $f \in R_\rho \backslash A_\rho$.

For the proof of Theorem 5.3.1, we need

Lemma 5.3.1 *Let $g(z) = \sum^\infty_{k=0} a_k z^k \in R_\rho(\rho > 1), l$ be any positive integer and $w_s(z) := \prod^s_{\nu=1}(z - \eta_\nu) = \sum^s_{k=0} C_k z^k$, where $\{\eta_\nu\}^s_{\nu=1}$ are any given s points in $|z| > \rho$ (or in $|z| < \rho$) , then*

$$H^{p_\nu - 1}_{l,r}(\eta_\nu; w_s g) < K^1_{l,r}(|\eta|, \rho), \qquad \nu = 1, \ldots, s \tag{5.3.1}$$

iff there is a $\rho_0 > \rho$ such that for $\nu = 1, 2, \ldots, s$

$$a_{(l+r)n-\nu} = \mathcal{O}\left(\rho_0^{-(l+r)n}\right) \left(or\ a_{(l+r-1)n-\nu} = \mathcal{O}(\rho_0^{-(l+r-1)n})\right).$$

proof : From (5.1.7)

$$\Delta_{rn-1,l}(z,g)=\sum_{k=0}^{n-1}\sum_{j=l}^{\infty}\beta_{j,r}(z^n)a_{k+(r+j-1)n}z^k.$$

Similarly, for any positive integer ν, we have

$$\Delta_{rn-1,l}(z,z^\nu g)=\sum_{k=0}^{n-1}\sum_{j=l}^{\infty}\beta_{j,r}(z^n)a_{k-\nu+(r+j-1)n}z^k.$$

According to the linearity property of $\Delta_{rn-1,l}(z;f)$ it follows that

$$\begin{aligned}
\Delta_{rn-1,l}(z,\omega_s g) &= \sum_{\nu=0}^{s}C_\nu\sum_{k=0}^{n-1}\sum_{j=l}^{\infty}\beta_{j,r}(z^n)a_{k-\nu+(r+j-1)n}z^k\\
&= \sum_{\nu=0}^{s}C_\nu\sum_{k=-\nu}^{n-\nu-1}\sum_{j=l}^{\infty}\beta_{j,r}(z^n)a_{k+(r+j-1)n}z^{k+\nu}\\
&= \sum_{j=l}^{\infty}\beta_{j,r}(z^n)\sum_{\nu=0}^{s}C_\nu\left(\sum_{k=-\nu}^{-1}+\sum_{k=0}^{n-1}-\sum_{k=n-\nu}^{n-1}\right)a_{k+(r+j-1)n}z^{k+\nu}\\
&= \sum_{j=l}^{\infty}\beta_{j,r}(z^n)\sum_{\nu=1}^{s}C_\nu\sum_{k=0}^{\nu-1}a_{k-\nu+(r+j-1)n}z^k+\omega_s(z)\Delta_{rn-1,l}(z;f)\\
&\quad -z^n\sum_{j=l}^{\infty}\beta_{j,r}(z^n)\sum_{\nu=1}^{s}C_\nu\sum_{k=0}^{\nu-1}a_{k+n-\nu+(r+j-1)n}z^k. \qquad (5.3.2)
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\sum_{j=l}^{\infty}\beta_{j,r}(z^n)\sum_{\nu=1}^{s}C_\nu\sum_{k=0}^{\nu-1}a_{k-\nu+(r+j-1)n}z^k\\
&=\sum_{j=l}^{\infty}\beta_{j,r}(z^n)\sum_{k=0}^{s-1}z^k\sum_{\nu=k+1}^{s}C_\nu a_{k-\nu+(r+j-1)n}\\
&=\sum_{k=0}^{s-1}z^k\sum_{\nu=1}^{s-k}C_{\nu-k}\sum_{j=l}^{\infty}\beta_{j,r}(z^n)a_{(r+j-1)n-\nu}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\sum_{j=l}^{\infty}\beta_{j,r}(z^n)\sum_{\nu=1}^{s}C_\nu\sum_{k=0}^{\nu-1}a_{k+n-\nu+(r+j-1)n}z^k\\
&=\sum_{j=l}^{\infty}\beta_{j,r}(z^n)\sum_{k=0}^{s-1}z^k\sum_{\nu=k+1}^{s}C_\nu a_{k+n-\nu+(r+j-1)n}\\
&=\sum_{k=0}^{s-1}z^k\sum_{\nu=1}^{s-k}C_{\nu+k}\sum_{j=l}^{\infty}\beta_{j,r}(z^n)a_{n+(r+j-1)n-\nu}.
\end{aligned}$$

Substituting these in (5.3.2) we have

$$\begin{aligned}
\Delta_{rn-1,l}(z,\omega_s g) &= \sum_{k=0}^{s-1}z^k\sum_{\nu=1}^{s-k}C_{\nu+k}\sum_{j=l}^{\infty}\beta_{j,r}(z^n)a_{(r+j-1)n-\nu}+\omega_s(z)\Delta_{rn-1,l}(z;g)\\
&\quad -z^n\sum_{k=0}^{s-1}z^k\sum_{\nu=1}^{s-k}C_{\nu+k}\sum_{j=l}^{\infty}\beta_{j,r}(z^n)a_{n+(r+j-1)n-\nu}.
\end{aligned}$$

With (5.2.2) this gives

$$\begin{aligned}\Delta_{rn-1,l}(z,\omega_s g) &= \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu} + \omega_s(z)\Delta_{rn-1,l}(z;g)\\ &\quad -z^n \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu}. \qquad (5.3.3)\end{aligned}$$

Now, since η_ν occurs p_ν times in $\{\eta_j\}_{j=1}^{\nu}$ hence $\omega^{(r)}(z) = 0$ at $z = \eta_\nu$ and $r = 0,\ldots,p_{\nu-1}$. Thus,

$$\begin{aligned}\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s g) &= 0 + \sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} z^{k-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{0,r}(j) a_{(r+j-1)n-\nu}\\ &\quad + \sum_{k=0}^{s-1}\sum_{j=l}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_{p_\nu-1} z^{k+n\lambda-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} a_{(r+j-1)n-\nu}\\ &\quad - \sum_{k=0}^{s-1}\sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j)(k+n+n\lambda)_{p_\nu-1} z^{k+n+n\lambda-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} a_{(r+j-1)n+n-\nu}.\end{aligned}$$

(5.3.4)

If points are in $|z| > \rho$ then

$$\begin{aligned}\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s g) &= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(r+l-1)n}} + \frac{|\eta_\nu|^{n(r-1)}}{(\rho-\epsilon)^{(r+l-1)n}} + \frac{|\eta_\nu|^{n+n(r-1)}}{\rho_0^{(l+r)n}}\right)\\ &= \mathcal{O}\left(\frac{|\eta_\nu|^{n(r-1)}}{(\rho-\epsilon)^{(r+l-1)n}} + \frac{|\eta_\nu|^{n+n(r-1)}}{\rho_0^{(l+r)n}}\right).\end{aligned}$$

Now for $|\eta_\nu| > \rho$, for a given $\epsilon > 0$ we can find $\eta > 0$ such that

$$\frac{|\eta_\nu|^{n+n(r-1)}}{\rho_0^{(l+r)n}} < \left(\frac{|\eta_\nu|}{\rho^{(1+l/r)}} - \eta\right)^{nr}, \rho_0 > \rho$$

and following the same steps as in 2.3.9 we have

$$\frac{|\eta_\nu|^{n(r-1)}}{(\rho-\epsilon)^{(r+l-1)n}} < \left(\frac{|\eta_\nu|}{\rho^{(1+l/r)}} - \eta\right)^{nr}, \qquad |\eta_\nu| > \rho.$$

Thus,

$$\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s g) = \mathcal{O}\left(\frac{|\eta_\nu|}{\rho^{(1+l/r)}} - \eta\right)^{nr}.$$

Hence

$$H_{l,r}^{p_\nu-1}(\eta_\nu; w_s g) < \frac{|\eta_\nu|}{\rho^{(1+l/r)}}, \qquad \nu = 1,2,\ldots,s.$$

Similarly if points are in $1 < |z| < \rho$ then from (5.3.4) we have

$$\begin{aligned}\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s g) &= \mathcal{O}\left(\frac{1}{\rho_0^{(r+l-1)n}} + \frac{|\eta_\nu|^{n(r-1)}}{\rho_0^{(r+l-1)n}} + \frac{|\eta_\nu|^{n+n(r-1)}}{(\rho-\epsilon)^{(l+r)n}}\right)\\ &= \mathcal{O}\left(\frac{|\eta_\nu|^{n(r-1)}}{\rho_0^{(r+l-1)n}} + \frac{|\eta_\nu|^{n+n(r-1)}}{(\rho-\epsilon)^{(l+r)n}}\right).\end{aligned}$$

Now for $1 < |\eta_\nu| < \rho$, for a given $\epsilon > 0$ we can find $\eta > 0$ such that

$$\frac{|\eta_\nu|^{n(r-1)}}{\rho_0^{(l+r-1)n}} < \left(\frac{|\eta_\nu|^{1-1/r}}{\rho^{(1+(l-1)/r)}} - \eta\right)^{nr}, \rho_0 > \rho$$

and following steps as for 2.3.9 we have

$$\frac{|\eta_\nu|^{n+n(r-1)}}{(\rho-\epsilon)^{(r+l)n}} < \left(\frac{|\eta_\nu|^{1-1/r}}{\rho^{(1+(l-1)/r)}} - \eta\right)^{nr}, \qquad |\eta_\nu| < \rho.$$

Thus,

$$\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s g) = \mathcal{O}\left(\frac{|\eta_\nu|^{1-1/r}}{\rho^{(1+(l-1)/r)}} - \eta\right)^{nr}.$$

Hence

$$H_{l,r}^{p_\nu-1}(\eta_\nu; w_s g) < \frac{|\eta_\nu|^{(1-1/r)}}{\rho^{1+(l-1)/r}}, \qquad \nu = 1, 2, \ldots, s.$$

Similarly if points are in $|z| < 1 < \rho$ then from (5.3.4) we have

$$\begin{aligned}
\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s g) &= \mathcal{O}\left(\frac{1}{\rho_0^{(r+l-1)n}} + \frac{1}{\rho_0^{(r+l-1)n}} + \frac{|\eta_\nu|^n}{(\rho-\epsilon)^{(l+r)n}}\right) \\
&= \mathcal{O}\left(\frac{1}{\rho_0^{(r+l-1)n}} + \frac{|\eta_\nu|^n}{(\rho-\epsilon)^{(l+r)n}}\right) \\
&= \mathcal{O}\left(\frac{1}{\rho_0^{(r+l-1)n}} + \frac{1}{(\rho-\epsilon)^{(l+r)n}}\right).
\end{aligned}$$

Now for $|\eta_\nu| < 1 < \rho$, for a given $\epsilon > 0$ we can find $\eta > 0$ such that

$$\frac{1}{\rho_0^{(l+r-1)n}} < \left(\frac{1}{\rho^{(1+(l-1)/r)}} - \eta\right)^{nr}, \rho_0 > \rho$$

and following steps as for 2.3.22

$$\frac{1}{(\rho-\epsilon)^{(r+l)n}} < \left(\frac{1}{\rho^{(1+(l-1)/r)}} - \eta\right)^{nr}, \qquad |\eta_\nu| < \rho.$$

Thus,

$$\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s g) = \mathcal{O}\left(\frac{1}{\rho^{(1+(l-1)/r)}} - \eta\right)^{nr}.$$

Hence

$$H_{l,r}^{p_\nu-1}(\eta_\nu; w_s g) < \frac{1}{\rho^{1+(l-1)/r}}, \qquad \nu = 1, 2, \ldots, s.$$

Conversely, suppose (5.3.1) is valid. Since $g \in R_\rho$, by continuity there is a $\rho_1 > \rho$ with

$$\rho < \rho_1 < min\left[\rho^{((l+r)+1)/(l+r)}, (\rho^{l+r-1} min_{1\leq\nu\leq s}|\eta_\nu|)^{1/(l+r)}\right] \tag{5.3.5}$$

such that

$$H_{l,r}^{p_\nu-1}(\eta_\nu; w_s g) < K_{l,r}^1(|\eta_\nu|, \rho_1), \qquad \nu = 1, \ldots, s. \tag{5.3.6}$$

From (5.3.3)

$$\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu} z^n$$

$$= -\Delta_{rn-1,l}(z, \omega_s g) + \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu}$$

$$+\omega_s(z)\Delta_{rn-1,l}(z; g)$$

or,

$$\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) z^{n(r-1)} a_{n+(r+j-1)n-\nu} z^n$$

$$= -\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-2} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu} z^n$$

$$-\Delta_{rn-1,l}(z, \omega_s g) + \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu}$$

$$+\omega_s(z)\Delta_{rn-1,l}(z; g)$$

or,

$$\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) a_{n+(r+j-1)n-\nu}$$

$$= -z^{-n(r-1)} \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-2} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu}$$

$$-z^{-n-n(r-1)}\Delta_{rn-1,l}(z, \omega_s g) + z^{-n-n(r-1)} \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{(r-j-1)n-\nu}$$

$$+z^{-n-n(r-1)}\omega_s(z)\Delta_{rn-1,l}(z; g).$$

Hence on taking t^{th} derivative we have

$$\sum_{k=t}^{s-1} (k)_t z^{k-t} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) a_{(r+j-1)n+n-\nu}$$

$$= -\sum_{b=0}^{t} (\binom{t}{b})(z^{-n-n(r-1)})^{(b)} \left(\sum_{k=0}^{s-1} \sum_{\lambda=0}^{r-2} C_{\lambda,r}(j) z^{k+n\lambda} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} a_{(r+j-1)n-n-\nu} \right)^{(t-b)} +$$

$$+\sum_{b=0}^{t} (\binom{t}{b}) \left(\frac{\omega_s(z)}{z^{n+n(r-1)}} \right)^{(b)} \Delta^{(t-b)}_{rn-1,l}(z; g) - \sum_{b=0}^{t} (\binom{t}{b})(z^{-n-n(r-1)})^{(b)} \Delta^{(t-b)}_{rn-1,l}(z; \omega_s g)$$

$$+\sum_{b=0}^{t} (\binom{t}{b})(z^{-n-n(r-1)})^{(b)} \left(\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu} \right)^{(t-b)} . \qquad (5.3.7)$$

On taking $t = p_\nu - 1$ and $z = \eta_\nu (\nu = 1, \dots, s)$, from (5.3.6) and the fact that $f \in R_\rho$ and the definition of p_ν we have

$$\sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} {\eta_\nu}^{k-p_\nu+1} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) a_{(r+j-1)n+n-\nu}$$

$$= \mathcal{O}\left(n^{p_\nu-1} |\eta_\nu|^{-n-n(r-1)} \frac{|\eta_\nu|^{n(r-2)}}{(\rho-\epsilon)^{n(r+l)}} \right)$$

$$+0 + \mathcal{O}\left(n^{p_\nu-1} |\eta_\nu|^{-n-n(r-1)} (K^1_{l,r}(|\eta_\nu|, \rho_1))^{rn} \right) + \mathcal{O}\left(n^{p_\nu-1} |\eta_\nu|^{-n-n(r-1)} \frac{|\eta_\nu|^{n(r-1)}}{(\rho-\epsilon)^{(r+l-1)n}} \right)$$

$$= \mathcal{O}\left(n^{p_\nu-1} max \left(\frac{|\eta_\nu|^{-2n}}{(\rho-\epsilon)^{(l+r)n}}, |\eta_\nu|^{-n} \frac{|\eta_\nu|^n}{\rho_1^{(l+r)n}}, \frac{|\eta_\nu|^{-n}}{(\rho-\epsilon)^{(l+r-1)n}} \right) \right)$$

$$= \mathcal{O}\left(n^{p_\nu-1} max \left(\frac{1}{\rho_1^{(l+r)n}}, \frac{1}{min|\eta_\nu|^n (\rho-\epsilon)^{(l+r-1)n}} \right) \right).$$

Since ϵ is arbitrary small hence by the choice of ρ_1 from (5.3.5) we have

$$\sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} {\eta_\nu}^{k-p_\nu+1} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) a_{(r+j-1)n+n-\nu}$$

$$= \mathcal{O}\left(n^{p_\nu-1} {\rho_1}^{-(l+r)n} \right), \qquad \nu = 1, 2, \dots, s. \tag{5.3.8}$$

Since η_ν are all distinct thus on solving (5.3.8) we have

$$\sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) a_{(r+j-1)n+n-\nu} = \mathcal{O}\left(n^\tau {\rho_1}^{-(l+r)n} \right), \tag{5.3.9}$$

$\tau = \max_{1 \le \nu \le s}[p_\nu - 1], k = 0, 1, \dots, s-1$. solving (5.3.9) we have

$$\sum_{j=l}^{\infty} C_{r-1,r}(j) a_{(r+j-1)n+n-\nu} = \mathcal{O}\left(n^\tau {\rho_1}^{-(l+r)n} \right), \qquad \nu = 1, 2, \dots, s,$$

so that by the choice of ρ_1

$$\begin{aligned}
a_{(r+l-1)n+n-\nu} &= \mathcal{O}\left(n^\tau {\rho_1}^{-(l+r)n} \right) - \sum_{j=l+1}^{\infty} \frac{C_{r-1,r}(j)}{C_{r-1,r}(l)} a_{(r+j-1)n+n-\nu} \\
&= \mathcal{O}\left(n^\tau {\rho_1}^{-(l+r)n} \right) + \mathcal{O}\left((\rho-\epsilon)^{-(r+l+1)n} \right) \\
&= \mathcal{O}\left(n^\tau {\rho_1}^{-(l+r)n} \right) \\
&= \mathcal{O}\left({\rho_0}^{-(l+r)n} \right)
\end{aligned}$$

where $\rho_0 \in (\rho, \rho_1)$.

Next, in the case when $\{\eta_\nu\}_{\nu=1}^s$ are in $1 < |z| < \rho$, suppose (5.2.8) is valid. Since $g \in R_\rho$, by continuity there is a $\rho_1 > \rho$ with

$$\rho < \rho_1 < min\left[\rho^{((l+r))/(l+r-1)}, (\rho^{l+r} min_{1 \le \nu \le s} |\eta_\nu|^{-1})^{1/(l+r-1)} \right.,$$

$$(\rho^{l+r-1} max_{1\le\nu\le s}|\eta_\nu|)^{1/(l+r-1)}\Big] \tag{5.3.10}$$

such that

$$H_{l,r}^{p_\nu-1}(\eta_\nu; w_s g) < K_{l,r}^1(|\eta_\nu|,\rho_1), \qquad \nu = 1,\ldots,s. \tag{5.3.11}$$

From (5.3.3)

$$\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu}$$

$$= -\Delta_{rn-1,l}(z,\omega_s g) + \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu} z^n$$

$$+\omega_s(z)\Delta_{rn-1,l}(z;g)$$

or,

$$\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) z^{n(r-1)} a_{(r+j-1)n-\nu}$$

$$= -\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-2} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu}$$

$$-\Delta_{rn-1,l}(z,\omega_s g) + \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu} z^n$$

$$+\omega_s(z)\Delta_{rn-1,l}(z;g)$$

or,

$$\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) a_{(r+j-1)n-\nu}$$

$$= -z^{-n(r-1)} \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-2} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu}$$

$$-z^{-n(r-1)}\Delta_{rn-1,l}(z,\omega_s g) + z^{-n(r-1)} \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu} z^n$$

$$+z^{-n(r-1)}\omega_s(z)\Delta_{rn-1,l}(z;g).$$

Hence on taking t^{th} derivative we have

$$\sum_{k=t}^{s-1} (k)_t z^{k-t} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{r-1,r}(j) a_{(r+j-1)n-\nu}$$

$$= -\sum_{b=0}^{t}(\binom{t}{b})(z^{-n(r-1)})^{(b)} \left(\sum_{k=0}^{s-1}\sum_{\lambda=0}^{r-2} C_{\lambda,r}(j) z^{k+n\lambda} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} a_{(r+j-1)n-\nu}\right)^{(t-b)} +$$

$$+\sum_{b=0}^{t}(\binom{t}{b})\left(\frac{\omega_s(z)}{z^{n(r-1)}}\right)^{(b)} \Delta_{rn-1,l}^{(t-b)}(z;g) - \sum_{b=0}^{t}(\binom{t}{b})(z^{-n(r-1)})^{(b)} \Delta_{rn-1,l}^{(t-b)}(z;\omega_s g)$$

$$+\sum_{b=0}^{t}(\binom{t}{b})(z^{-n(r-1)})^{(b)}\left(\sum_{k=0}^{s-1}z^k\sum_{\nu=1}^{s-k}C_{\nu+k}\sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1}C_{\lambda,r}(j)z^{n\lambda}a_{n+(r+j-1)n-\nu}z^n\right)^{(t-b)}. \quad (5.3.12)$$

On taking $t=p_\nu-1$ and $z=\eta_\nu(\nu=1,\ldots,s)$, from (5.3.11) and the fact that $f\in R_\rho$ and the definition of p_ν we have

$$\sum_{k=p_\nu-1}^{s-1}(k)_{p_\nu-1}{\eta_\nu}^{k-p_\nu+1}\sum_{\nu=1}^{s-k}C_{\nu+k}\sum_{j=l}^{\infty}C_{r-1,r}(j)a_{(r+j-1)n-\nu}$$

$$=\mathcal{O}\left(n^{p_\nu-1}|\eta_\nu|^{-n(r-1)}\frac{|\eta_\nu|^{n(r-2)}}{(\rho-\epsilon)^{n(r+l-1)}}\right)$$
$$+0+\mathcal{O}\left(n^{p_\nu-1}|\eta_\nu|^{-n(r-1)}(K^1_{l,r}(|\eta_\nu|,\rho_1))^{rn}\right)+\mathcal{O}\left(n^{p_\nu-1}|\eta_\nu|^{-n(r-1)}\frac{|\eta_\nu|^{n(r-1)+n}}{(\rho-\epsilon)^{(r+l)n}}\right)$$
$$=\mathcal{O}\left(n^{p_\nu-1}max\left(\frac{|\eta_\nu|^{-n}}{(\rho-\epsilon)^{(l+r-1)n}},|\eta_\nu|^{-n(r-1)}\frac{|\eta_\nu|^{n(r-1)}}{\rho_1^{(l+r-1)n}},\frac{|\eta_\nu|^{n}}{(\rho-\epsilon)^{(l+r)n}}\right)\right).$$

Since ϵ is arbitrary small hence by the choice of ρ_1 from (5.3.10)we have

$$\sum_{k=p_\nu-1}^{s-1}(k)_{p_\nu-1}{\eta_\nu}^{k-p_\nu+1}\sum_{\nu=1}^{s-k}C_{\nu+k}\sum_{j=l}^{\infty}C_{r-1,r}(j)a_{(r+j-1)n-\nu}$$
$$=\mathcal{O}\left(n^{p_\nu-1}\rho_1^{-(l+r-1)n}\right),\qquad \nu=1,2,\ldots,s. \quad (5.3.13)$$

Since η_ν are all distinct thus on solving (5.3.13) we have

$$\sum_{\nu=1}^{s-k}C_{\nu+k}\sum_{j=l}^{\infty}C_{r-1,r}(j)a_{(r+j-1)n-\nu}=\mathcal{O}\left(n^\tau\rho_1^{-(l+r-1)n}\right), \quad (5.3.14)$$

$\tau=\max_{1\le\nu\le s}[p_\nu-1], k=0,1,\ldots,s-1$. solving (5.3.14) we have

$$\sum_{j=l}^{\infty}C_{r-1,r}(j)a_{(r+j-1)n-\nu}=\mathcal{O}\left(n^\tau\rho_1^{-(l+r-1)n}\right),\qquad \nu=1,2,\ldots,s,$$

so that by the choice of ρ_1

$$\begin{aligned}a_{(r+l-1)n-\nu} &= \mathcal{O}\left(n^\tau\rho_1^{-(l+r-1)n}\right)-\sum_{j=l+1}^{\infty}\frac{C_{r-1,r}(j)}{C_{r-1,r}(l)}a_{(r+j-1)n-\nu}\\ &= \mathcal{O}\left(n^\tau\rho_1^{-(l+r-1)n}\right)+\mathcal{O}\left((\rho-\epsilon)^{-(r+l)n}\right)\\ &= \mathcal{O}\left(n^\tau\rho_1^{-(l+r-1)n}\right)\\ &= \mathcal{O}\left(\rho_0^{-(l+r-1)n}\right)\end{aligned}$$

where $\rho_0\in(\rho,\rho_1)$.

Finally, for the case when $\{\eta_\nu\}_{\nu=1}^s$ are in $|z|<1<\rho$, suppose (5.2.8) is valid. Since $g\in R_\rho$, by continuity there is a $\rho_1>\rho$ with

$$\rho<\rho_1<min\left[\rho^{((l+r))/(l+r-1)},(\rho^{l+r-1}min_{1\le\nu\le s}|\eta_\nu|^{-1})^{1/(l+r-1)}\right] \quad (5.3.15)$$

such that

$$H_{l,r}^{p_\nu-1}(\eta_\nu;w_sg) < K_{l,r}^1(|\eta_\nu|,\rho_1), \qquad \nu=1,\ldots,s. \tag{5.3.16}$$

From (5.3.3)

$$\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu}$$

$$= -\Delta_{rn-1,l}(z,\omega_s g) + \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu} z^n$$

$$+\omega_s(z)\Delta_{rn-1,l}(z;g)$$

or,

$$\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{0,r}(j) a_{(r+j-1)n-\nu}$$

$$= -\sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{(r+j-1)n-\nu}$$

$$-\Delta_{rn-1,l}(z,\omega_s g) + \sum_{k=0}^{s-1} z^k \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j) z^{n\lambda} a_{n+(r+j-1)n-\nu} z^n$$

$$+\omega_s(z)\Delta_{rn-1,l}(z;g).$$

Hence on taking t^{th} derivative we have

$$\sum_{k=t}^{s-1} (k)_t z^{k-t} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{0,r}(j) a_{(r+j-1)n-\nu}$$

$$= \sum_{k=0}^{s-1}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_t z^{k+n\lambda-t} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} a_{(r+j-1)n-\nu} +$$

$$+\sum_{b=0}^{t} (\binom{t}{b}) (\omega_s(z))^{(b)} \Delta_{rn-1,l}^{(t-b)}(z;g) - \Delta_{rn-1,l}^{(t)}(z;\omega_s g)$$

$$+\sum_{k=0}^{s-1}\sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} (k+n\lambda+n)_t C_{\lambda,r}(j) z^{k+n\lambda+n-t} a_{n+(r+j-1)n-\nu}. \tag{5.3.17}$$

On taking $t = p_\nu - 1$ and $z = \eta_\nu (\nu = 1,\ldots,s)$, from (5.3.16) and the fact that $f \in R_\rho$ and the definition of p_ν we have

$$\sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} {\eta_\nu}^{k-p_\nu+1} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{0,r}(j) a_{(r+j-1)n-\nu}$$

$$= \mathcal{O}\left(n^{p_\nu-1}\frac{|\eta_\nu|^n}{(\rho-\epsilon)^{n(r+l-1)}}\right)$$

$$+0+\mathcal{O}\left(n^{p_\nu-1}(K_{l,r}^1(|\eta_\nu|,\rho_1))^{rn}\right)+\mathcal{O}\left(n^{p_\nu-1}\frac{1}{(\rho-\epsilon)^{(r+l)n}}\right)$$

$$= \mathcal{O}\left(n^{p_\nu - 1} max\left(\frac{|\eta_\nu|^n}{(\rho-\epsilon)^{(l+r-1)n}}, \frac{1}{\rho_1^{(l+r-1)n}}, \frac{1}{(\rho-\epsilon)^{(l+r)n}}\right)\right).$$

Since ϵ is arbitrary small hence by the choice of ρ_1 from (5.3.15) we have

$$\sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} \eta_\nu{}^{k-p_\nu+1} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{0,r}(j) a_{(r+j-1)n-\nu}$$

$$= \mathcal{O}\left(n^{p_\nu-1} \rho_1{}^{-(l+r-1)n}\right), \qquad \nu = 1, 2, \ldots, s. \tag{5.3.18}$$

Since η_ν are all distinct thus on solving (5.3.18) we have

$$\sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{0,r}(j) a_{(r+j-1)n-\nu} = \mathcal{O}\left(n^\tau \rho_1^{-(l+r-1)n}\right), \tag{5.3.19}$$

$\tau = \max_{1\le\nu\le s}[p_\nu - 1], k = 0, 1, \ldots, s-1$. solving (5.3.19) we have

$$\sum_{j=l}^{\infty} C_{0,r}(j) a_{(r+j-1)n-\nu} = \mathcal{O}\left(n^\tau \rho_1^{-(l+r-1)n}\right), \qquad \nu = 1, 2, \ldots, s,$$

so that by the choice of ρ_1

$$\begin{aligned}
a_{(r+l-1)n-\nu} &= \mathcal{O}\left(n^\tau \rho_1^{-(l+r-1)n}\right) - \sum_{j=l+1}^{\infty} \frac{C_{0,r}(j)}{C_{0,r}(l)} a_{(r+j-1)n-\nu} \\
&= \mathcal{O}\left(n^\tau \rho_1^{-(l+r-1)n}\right) + \mathcal{O}\left((\rho-\epsilon)^{-(r+l)n}\right) \\
&= \mathcal{O}\left(n^\tau \rho_1^{-(l+r-1)n}\right) \\
&= \mathcal{O}\left(\rho_0{}^{-(l+r-1)n}\right)
\end{aligned}$$

where $\rho_0 \in (\rho, \rho_1)$.

Proof of Theorem 5.3.1 : For points $\{\eta_\nu\}_{\nu=1}^s$ in $|z| > \rho$ let

$$g(z) = \frac{f(z)}{\prod_{\nu=1}^{l+r}(z-\eta_\nu)} = \sum_{k=0}^{\infty} a_k z^k, \tag{5.3.20}$$

thus $f(z) = w_{l+r}(z) g(z)$ and $g \in R_\rho$. According to Lemma 5.3.1, there is a $\rho_0 > \rho$ such that

$$a_{(l+r)n-\nu} = \mathcal{O}\left(\rho_0{}^{-(l+r)n}\right) \qquad \nu = 1, 2, \ldots, l+r,$$

so that

$$\overline{\lim_{k\to\infty}} |a_k|^{1/k} \le \frac{1}{\rho_0} < \frac{1}{\rho},$$

hence, $g \in R_\rho \backslash A_\rho$ which gives $f \in R_\rho \backslash A_\rho$.

If $\{\eta_\nu\}_{\nu=1}^{s}$ are in $|z| < \rho$, then we set

$$g(z) = [f(z) - L_{l+r-2}(z)] / \prod_{\nu=1}^{l+r-1} (z - \eta_\nu) = \sum_{k=0}^{\infty} a_k z^k,$$

where $L_{l+r-2}(z)$ is the Lagrange interpolating polynomial of $f(z)$ of degree $l + r - 2$ at $\{\eta_\nu\}_{\nu=1}^{l+r-1}$, then we have $f(z) = \omega_{l+r-1}(z)g(z) + L_{l+r-2}(z)$ where $g(z) \in R_\rho$ and $L_{l-r-2}(z) \in R_\rho \backslash A_\rho$. Similarly as in above case we can show that $g \in R_\rho \backslash A_\rho$ so that $f \in R_\rho \backslash A_\rho$.

Remark 5.3.1 For $p_\nu = 1, \forall \nu$ Theorem 5.3.1 gives Corollary 1 [20].

Remark 5.3.2 For $r = 1$ Theorem 5.3.1 gives Theoren 2.1.7.

Theorem 5.3.2 *suppose $f \in A_\rho (\rho > 1), l$ is a positive integer, then*

(a) there are at most $l + r - 1$ points $\{\eta_\nu\}_{\nu=1}^{l+r-1}$ in $|z| > \rho$ with

$$H_{l,r}^{p_\nu - 1}(\eta_\nu; f) < K_{l,r}^1(|\eta_\nu|, \rho), \qquad \nu = 1, \ldots, l + r - 1$$

(b) there are at most $l + r - 2$ points $\{\eta_\nu\}_{\nu=1}^{l+r-2}$ in $|z| < \rho$ with

$$H_{l,r}^{p_\nu - 1}(\eta_\nu; f) < K_{l,r}^1(|\eta_\nu|, \rho), \qquad \nu = 1, \ldots, lm - 1.$$

(Points can be in $1 < |z| < \rho$ or $|z| < 1 < \rho$.)

(c) The degree of (l, r, ρ) - distinguished point of $f(z)$ is neither greater than $l + r - 1$ in $|z| > \rho$ nor greater than $l + r - 2$ in $|z| < \rho$.

(d) If either z is in $|z| > \rho$ and $t \geq l + r$ or z is in $|z| < \rho$ and $t \geq l + r - 1$, then

$$\overline{\lim_{n \to \infty}} \left[\sum_{\nu=0}^{t} |\Delta_{rn-1,l}^{(\nu)}(z; f)| \right]^{1/rn} = K_{l,r}^1(|z|, \rho).$$

Moreover, for given any η in $|z| > \rho$ and $0 \leq t < l + r$ or for η in $|z| < \rho$ and $0 \leq t < l + r - 1$, there is an $f \in A_\rho$ for which

$$\overline{\lim_{n \to \infty}} \left[\sum_{\nu=0}^{t} |\Delta_{rn-1,l}^{(\nu)}(\eta; f)| \right]^{1/rn} < K_{l,r}^1(|\eta|, \rho).$$

Remark 5.3.3 For $p_\nu = 1, \forall \nu$ (a) and (b) of Theorem 5.3.2 gives Corollary 2 [20].

Remark 5.3.4 For $r = 1$ Theorem 5.3.2 gives Theoren 3 [28].

Clearly Theorem 5.3.2 follows from Theorem 5.3.1 excluding second part of (d) which follows from the following Theorem 5.3.3.

Theorem 5.3.3 *Let $f \in A_\rho(\rho > 1), l$ be any positive integer and $\{\eta_\nu\}_{\nu=1}^s$ be any s points in $|z| > \rho, s \le l+r-1$ (or in $|z| < \rho, s \le l+r-2$) , with the numbers p_ν of the appearence of η_ν in $\{\eta_j\}_{j=1}^\nu$. Then the neccessary and sufficient condition for*

$$H_{l,r}^{p_\nu-1}(\eta_\nu; f) < K_{l,r}^1(|\eta_\nu|, \rho), \qquad \nu = 1, \dots, s \tag{5.3.21}$$

is

$$f(z) = w_s(z)G_s(z) + G_0(z) \tag{5.3.22}$$

where $w_s(z) := \prod_{j=1}^s (z-\eta_j)$, $G_0 \in R_\rho \backslash A_\rho$ and $G_s(z) = \sum_{j=0}^\infty \alpha_j z^j \in A_\rho$ with

$$\alpha_{(l+r)n-\nu} = 0 (or, \alpha_{(l+r-1)n-\nu} = 0), \qquad \nu = 1, 2, \dots, s.$$

proof : *Sufficiency.* Suppose $f(z)$ can be expressed as (5.3.22). Since $G_0 \in R_\rho \backslash A_\rho$, according to Theorem 5.2.1

$$\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu; G_0) < \mathcal{O}\left([K_{l,r}^1(|\eta_\nu|, \rho)]^n\right).$$

that is there exists a $\rho_1 > \rho$ such that

$$\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu; G_0) = \mathcal{O}\left([K_{l,r}^1(|\eta_\nu|, \rho_1)]^n\right). \tag{5.3.23}$$

If points are in $|z| > \rho$ then from (5.3.4) we have

$$\begin{aligned} \Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s G) &= 0 + \sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} z^{k-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l}^{\infty} C_{0,r}(j) \alpha_{(r+j-1)n-\nu} \\ &+ \sum_{k=0}^{s-1} \sum_{j=l}^{\infty} \sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_{p_\nu-1} z^{k+n\lambda-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \alpha_{(r+j-1)n-\nu} \\ &- \sum_{k=0}^{s-1} \sum_{j=l}^{\infty} \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j)(k+n+n\lambda)_{p_\nu-1} z^{k+n+n\lambda-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \alpha_{(r-j-1)n-n-\nu}. \end{aligned} \tag{5.3.24}$$

Now since from hypothesis $\alpha_{(l+r)n-\nu} = 0$ thus,

$$\begin{aligned} &\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_s G) \\ &= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(r+l-1)n}} + \frac{|\eta_\nu|^{n(r-1)}}{(\rho-\epsilon)^{(r+l-1)n}}\right) + \\ &+ \sum_{k=0}^{s-1} \sum_{j=l+1}^{\infty} \sum_{\lambda=0}^{r-1} C_{\lambda,r}(j)(k+n+n\lambda)_{p_\nu-1} z^{k+n+n\lambda-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \alpha_{(r+j-1)n+n-\nu} \end{aligned}$$

$$= \mathcal{O}\left(\frac{|\eta_\nu|^{n(r-1)}}{(\rho-\epsilon)^{(r+l-1)n}} + \frac{|\eta_\nu|^{n+n(r-1)}}{(\rho-\epsilon)^{(r+l+1)n}}\right). \tag{5.3.25}$$

By the arbitraryness of $\epsilon > 0$, from (5.3.22), (5.3.23) and (5.3.25) we have

$$\begin{aligned} H_{l,r}^{p_\nu-1}(\eta_\nu; f) &\leq max\left(K_{l,r}^1(|\eta_\nu|, \rho_1), \frac{|\eta_\nu|^{1-1/r}}{\rho^{1+(l-1)/r}}, \frac{|\eta_\nu|}{\rho^{1+(l+1)/r}}\right) \\ &= max\left(\frac{|\eta_\nu|}{\rho_1{}^{1+l/r}}, \frac{|\eta_\nu|^{1-1/r}}{\rho^{1+(l-1)/r}}, \frac{|\eta_\nu|}{\rho^{1+(l+1)/r}}\right) \\ &< \frac{|\eta_\nu|}{\rho^{1+l/r}} \\ &= K_{l,r}^1(|\eta_\nu|, \rho). \end{aligned}$$

If points are in $1 < |z| < \rho$ then from the hypothesis $\alpha_{(l+r-1)n-\nu} = 0$ thus from (5.3.24) we have

$$\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_sG) = 0 + \sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} z^{k-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l+1}^{\infty} C_{0,r}(j)\alpha_{(r+j-1)n-\nu}$$

$$+\sum_{k=0}^{s-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1} C_{\lambda,r}(j)(k+n\lambda)_{p_\nu-1} z^{k+n\lambda-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k}\alpha_{(r+j-1)n-\nu}$$

$$-\sum_{k=0}^{s-1}\sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1} C_{\lambda,r}(j)(k+n+n\lambda)_{p_\nu-1} z^{k+n+n\lambda-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k}\alpha_{(r+j-1)n+n-\nu}$$

$$= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(r+l)n}} + \frac{|\eta_\nu|^{n(r-1)}}{(\rho-\epsilon)^{(r+l)n}} + \frac{|\eta_\nu|^{n+n(r-1)}}{(\rho-\epsilon)^{(r+l)n}}\right)$$

$$= \mathcal{O}\left(\frac{|\eta_\nu|^{n(r-1)}}{(\rho-\epsilon)^{(r+l)n}} + \frac{|\eta_\nu|^{n+n(r-1)}}{(\rho-\epsilon)^{(r+l)n}}\right). \tag{5.3.26}$$

By the arbitraryness of $\epsilon > 0$, from (5.3.22), (5.3.23) and (5.3.26) we have

$$\begin{aligned} H_{l,r}^{p_\nu-1}(\eta_\nu; f) &\leq max\left(K_{l,r}^1(|\eta_\nu|, \rho_1), \frac{|\eta_\nu|^{1-1/r}}{\rho^{1+l/r}}, \frac{|\eta_\nu|}{\rho^{1+l/r}}\right) \\ &= max\left(\frac{|\eta_\nu|^{1-1/r}}{\rho_1{}^{1+(l-1)/r}}, \frac{|\eta_\nu|^{1-1/r}}{\rho^{1+l/r}}, \frac{|\eta_\nu|}{\rho^{1+l/r}}\right) \\ &< \frac{|\eta_\nu|^{1-1/r}}{\rho^{1+(l-1)/r}} \\ &= K_{l,r}^1(|\eta_\nu|, \rho). \end{aligned}$$

Similarly if points are in $|z| < 1 < \rho$ then from the hypothesis $\alpha_{(l+r-1)n-\nu} = 0$ thus from (5.3.24) we have

$$\Delta_{rn-1,l}^{(p_\nu-1)}(\eta_\nu, w_sG)$$

$$= 0 + \sum_{k=p_\nu-1}^{s-1} (k)_{p_\nu-1} z^{k-(p_\nu-1)} \sum_{\nu=1}^{s-k} C_{\nu+k} \sum_{j=l+1}^{\infty} C_{0,r}(j)\alpha_{(r+j-1)n-\nu}$$

$$+\sum_{k=0}^{s-1}\sum_{j=l+1}^{\infty}\sum_{\lambda=1}^{r-1}C_{\lambda,r}(j)(k+n\lambda)_{p_\nu-1}z^{k+n\lambda-(p_\nu-1)}\sum_{\nu=1}^{s-k}C_{\nu+k}\alpha_{(r+j-1)n-\nu}$$

$$-\sum_{k=0}^{s-1}\sum_{j=l}^{\infty}\sum_{\lambda=0}^{r-1}C_{\lambda,r}(j)(k+n+n\lambda)_{p_\nu-1}z^{k+n+n\lambda-(p_\nu-1)}\sum_{\nu=1}^{s-k}C_{\nu+k}\alpha_{(r+j-1)n+n-\nu}$$

$$=\mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(r+l)n}}+\frac{1}{(\rho-\epsilon)^{(r+l)n}}+\frac{|\eta_\nu|^n}{(\rho-\epsilon)^{(r+l)n}}\right)$$

$$=\mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(r+l)n}}+\frac{|\eta_\nu|^n}{(\rho-\epsilon)^{(r+l)n}}\right). \tag{5.3.27}$$

By the arbitraryness of $\epsilon > 0$, from (5.3.22), (5.3.23) and (5.3.27) we have

$$\begin{aligned}
H_{l,r}^{p_\nu-1}(\eta_\nu; f) &\leq max\left(K_{l,r}^1(|\eta_\nu|,\rho_1), \frac{1}{\rho^{1+l/r}}, \frac{|\eta_\nu|^{1/r}}{\rho^{1+l/r}}\right)\\
&= max\left(\frac{1}{\rho_1{}^{1+(l-1)/r}}, \frac{1}{\rho^{1+l/r}}, \frac{|\eta_\nu|^{1/r}}{\rho^{1+l/r}}\right)\\
&< \frac{1}{\rho^{1+(l-1)/r}}\\
&= K_{l,r}^1(|\eta_\nu|,\rho).
\end{aligned}$$

Neccessity. Suppose f satisfies (5.3.21). Let for $|z| > \rho$

$$g(z) = f(z)/w_s(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g \in A_\rho.$$

According to Lemma 5.3.1, from (5.3.21) there exists a $\rho_0 > \rho$ such that

$$a_{(l+r)n-\nu} = \mathcal{O}\left(\rho_0^{-(l+r)n}\right), \qquad \nu = 1, \ldots, s.$$

We set

$$\alpha_{(l+r)n-\nu} = \begin{cases} 0, \text{if } \ \nu = 1, \ldots, s; n = 1, 2, \ldots, \\ a_{(l+r)n-\nu}, \text{if } \ \nu = s+1, \ldots, l+r; n = 1, 2, \ldots, \end{cases}$$

and $G_s(z) = \sum_{j=0}^{\infty}\alpha_j z^j, g_0(z) = g(z) - G_s(z)$. Clearly $G_s \in A_\rho$ with $\alpha_{(l+r)n-\nu} = 0$, $(\nu = 1, \ldots, s)$ and $g_0(z) = \sum_{j=0}^{\infty}\gamma_j z^j$ with

$$\gamma_{(l+r)n-\nu} = \begin{cases} a_{(l+r)n-\nu}, \text{if } \ \nu = 1, \ldots, s; n = 1, 2, \ldots, \\ 0, \text{if } \ \nu = s+1, \ldots, l+r; n = 1, 2, \ldots, \end{cases}$$

hence $g_0 \in R_{\rho_0}$. Then we have

$$f(z) = w_s(z)g(z) = w_s(z)[G_s(z) + g_0(z)] = w_s(z)G_s(z) + G_0(z),$$

where $G_0(z) = w_s(z)g_0(z) \in R_{\rho_0}$ and since $\rho_0 > \rho$ thus $G_0(z) \in R_\rho \backslash A_\rho$.

In case $\{\eta_\nu\}_{\nu=1}^{s}$ are in $|z|<\rho$, the proof of sufficeincy is similar. For the necessity part we set

$$g(z)=[f(z)-L_{s-1}(z)]/w_s(z),$$

where $L_{s-1}(z)$ is the Lagrange interpolating polynomial of $f(z)$ of degree $s-1$ at $\{\eta_\nu\}_{\nu=1}^{s}$, then we have $f(z)=w_s(z)g(z)+L_{s-1}(z)$ where $g\in A_\rho$ and $L_{s-1}\in R_\rho\backslash A_\rho$. Similarly we can show that $g(z)=G_s(z)+g_0(z)$, where $g_0\in R_\rho\backslash A_\rho$ and $G_s\in A_\rho$ with $\alpha_{(l-r-1)n-\nu}=0, (\nu=1,2,\ldots,s)$ and obtain (5.3.22).

Corollary 5.3.1. *Let $f\in A_\rho, (\rho>1), l$ be any positive integer and $\{\eta_\nu\}_{\nu=1}^{s}$ be any s distinct points in $|z|>\rho, s\le l+r-1$ (or in $|z|<\rho, s\le l+r-2$). Then the necessary and sufficient condition for*

$$H_{l,r}(\eta_\nu;f)<K_{l,r}^{1}(|\eta_\nu|,\rho)\quad \nu=1,\ldots,s$$

is

$$f(z)=w_s(z)G_s(z)+G_0(z),$$

where $w_s(z), G_0(z)$ and $G_s(z)$ have the same meanings as in Theorem 5.3.3.

Corollary 5.3.2. *Let $f\in A_\rho, (\rho>1), l$ be any positive integer and η be any given point in $|z|>\rho, s\le l+r-1$ (or in $|z|<\rho, s\le l+r-2$). Then the necessary and sufficient condition for*

$$H_{l,r}^{\nu}(\eta;f)<K_{l,r}^{1}(|\eta|,\rho),\qquad \nu=1,\ldots,s-1$$

is

$$f(z)=(z-\eta)^sG_s(z)+G_0(z),$$

where $G_0(z)$ and $G_s(z)$ have the same meanings as in Theorem 5.3.3.

Remark 5.3.5 For $p_\nu=1,\forall\nu$ Theorem 5.3.3 gives Corollary 5.3.1 which generalizes Corollary 2 [20].

Remark 5.3.6 For $r=1$ Theorem 5.3.3 gives Theoren 2.1.8.

5.4 In this section we consider a set containing the points in $|z|<\rho$ and $|z|>\rho$ and generalize Theorem 5.1.2 for the case when the points $\{z_j\}_1^s$ can be coincided with each other. We call a set $Z=\{\eta_j\}_1^s$ with $|\eta_j|<\rho, j=1,\ldots,\mu$ and $|\eta_j|>\rho, j=\mu+1,\ldots,s$and

p_ν denots the number of appearence of η_ν in $\{\eta_j\}_{j=1}^{\nu}, \nu = 1, \ldots, s$. an (l, r, ρ)-distinguished set if there exists an $f \in A_\rho$ such that $H_{l,r}^{p_\nu - 1}(\eta_\nu; f) < K_{l,r}^{1}(|\eta_\nu|, \rho),\ \ \nu = 1, \ldots, s$. We define the matrices X, Y and M(X,Y) as follows:

$$X = \begin{pmatrix} 1 & z^{(p_1-1)}\big|_{z=\eta_1} & \cdots & (z^{l+r-2})^{(p_1-1)}\big|_{z=\eta_1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (z)^{(p_\mu-1)}\big|_{z=\eta_\mu} & \cdots & (z^{l+r-2})^{(p_\mu-1)}\big|_{z=\eta_\mu} \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & (z)^{(p_{\mu+1}-1)}\big|_{z=\eta_{\mu+1}} & \cdots & (z^{l+r-1})^{(p_{\mu+1}-1)}\big|_{z=\eta_{\mu+1}} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (z)^{(p_s-1)}\big|_{z=\eta_s} & \cdots & (z^{l+r-1})^{(p_s-1)}\big|_{z=\eta_s} \end{pmatrix}.$$

The matrices X and Y are of order $(\mu \times (l+r-1))$ and $(s-\mu) \times (l+r)$ respectively. Define

$$M = M(X,Y) = \begin{pmatrix} X & & & \\ & X & & 0 \\ & & \ddots & \\ & 0 & & X \\ Y & & & \\ & Y & & 0 \\ & & \ddots & \\ & 0 & & Y \end{pmatrix},$$

where X occurs $l+r$ times and Y occurs $l+r-1$ times beginning under the last X. The matrix M is of order $(s(l+r-1)+\mu) \times (l+r-1)(l+r)$. We now formulate

Theorem 5.4.1 *Suppose $Z = \{z_j\}_1^s$ is a set of points in C such that $|z_j| < \rho$ $(j = 1, \ldots, \mu)$ and $|z_j| > \rho$ $(j = \mu+1, \ldots, s)$ and $|z_j| \neq 1,\ j = 1, \ldots, s$, when $\beta_{l,r}(z)$ has a zero on the unit circle. Then the set Z is (l, r, ρ) distinguished iff*

$$rank\ \ M < (l+r-1)(l+r). \tag{5.4.1}$$

Before giving the proof we state Lemma 2 [20] which we shall be using later.

Lemma 5.4.1 (a) If $|z| > 1$ then there exist constants $C_1 = C_1(r) > 0$ and $N_1 = N_1(r, z)$ such that

$$C_1 j^{r-1} |z|^{n(r-1)} \leq |\beta_{j,r}(z^n)|, \qquad \text{for} \qquad n \geq N_1.$$

(b) If $|z| < 1$ then there are constants $C_2 = C_2(r) > 0$ and $N_2 = N_2(r, j, z)$ such that

$$C_2 j^{r-1} \leq |\beta_{j,r}(z^n)|, \qquad \text{for} \qquad n \geq N_2.$$

(c) If $\beta_{j,r}$ has no zero on the unit circle, then there is a constant $C_3 = C_3(j, r) > 0$ such that

$$C_3 \leq |\beta_{j,r}(z^n)|, \qquad \text{when} \qquad |z| = 1.$$

Proof of Theorem 5.4.1 : First suppose *rank* $M < (l+r-1)(l+r)$. Then there exists a non-zero vector $b = (b_0, b_1, \ldots, b_{(l+r-1)(l+r)-1})$ such that

$$M.b^T = 0. \tag{5.4.2}$$

Set

$$\begin{aligned} f(z) &= \sum_{N=0}^{\infty} a_N z^N \\ &= \left\{b_0 + b_1 z + \ldots + b_{(l+r-1)(l+r)-1} z^{(l+r-1)(l+r)-1}\right\} \left\{1 - \left(\frac{z}{\rho}\right)^{(l+r-1)(l-r)}\right\}^{-1}. \end{aligned}$$

Clearly $f \in A_\rho$ and that

$$a_N = b_k \rho^{-(l+r-1)(l+r)\nu} \tag{5.4.3}$$

where $N = (l+r-1)(l+r)\nu + k$, $k = 0, 1, \ldots, (l+r-1)(l+r) - 1, \nu = 0, 1, \ldots$.

From (5.4.2) and (5.4.3), we have

$$\left(\sum_{k=0}^{l+r-2} a_{(l+r-1)n+k} z_j^k\right)^{(p_j-1)} = 0 \qquad \text{for each n and } j = 1, 2, \ldots, \mu. \tag{5.4.4.}$$

and

$$\left(\sum_{k=0}^{l+r-1} a_{(l+r)n+k} z_j^k\right)^{(p_j-1)} = 0 \qquad \text{for each } n \text{ and } j = \mu+1, \ldots, s. \tag{5.4.5}$$

For any integer $n > 0$ let p and q be determined by

$$(l+r-1)n + q = (l+r)p, \qquad 0 \leq q < l+r$$

then for $j \geq \mu + 1$ from (5.4.5)

$$\begin{aligned} \left(\sum_{k=0}^{n-1} a_{k+(l+r-1)} z_j^k\right)^{(p_j-1)} &= \left(\sum_{k=0}^{q-1} a_{k+(l+r-1)n} z_j^k + \sum_{k=q}^{n-1} a_{k+(l+r-1)n} z_j^k\right)^{(p_j-1)} \\ &= \left(\sum_{k=0}^{q-1} a_{k+(l+r-1)n} z_j^k + \sum_{k=0}^{l+r-1} \sum_{\nu=p}^{n-1} a_{(l+r)\nu+k} z_j^{(l+r)\nu+k-(l+r-1)n}\right)^{(p_j-1)} \end{aligned}$$

$$= \left(\sum_{k=0}^{q-1} a_{k+(l+r-1)n} z_j^k + 0\right)^{(p_j-1)}$$

$$= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(l+r-1)n}}\right). \qquad \text{(for large } n)$$

This for $\mu < j \leq s$ giyes

$$\begin{aligned}
\Delta_{rn-1,l}^{(p_j-1)}(z_j; f) &= \left(\sum_{q=l}^{\infty}\sum_{k=0}^{n-1} \beta_{q,r}(z_j^n) a_{k+(r+q-1)n} z_j^k\right)^{(p_j-1)} \\
&= \left(\beta_{l,r}(z_j^n)\sum_{k=0}^{n-1} a_{k+(l+r-1)n} z_j^k + \right. \\
&\quad \left. \sum_{q=l+1}^{\infty}\sum_{k=0}^{n-1} \beta_{q,r}(z_j^n) a_{k+(r+q-1)n} z_j^k\right)^{(p_j-1)} \\
&= \mathcal{O}\left(\frac{|z_j|^{n(r-1)}}{(\rho-\epsilon)^{(r+l-1)n}}\right) + \mathcal{O}\left(\frac{|z_j|^{n+n(r-1)}}{(\rho-\epsilon)^{n(r+l)+n}}\right).
\end{aligned} \tag{5.4.6}$$

Now choose $\epsilon_1 > 0$ so small that

$$\frac{1}{(\rho-\epsilon_1)^{(l+r-1)}} < \frac{|z_j|}{\rho^{l+r}} \quad ; \quad |z_j| > \rho$$

so

$$0 < \frac{|z_j|}{\rho^{l+r}} - \frac{1}{(\rho-\epsilon_1)^{l+r-1}}.$$

Choose $\eta_1 > 0$ such that

$$0 < \eta_1 < \frac{|z_j|}{\rho^{l+r}} - \frac{1}{(\rho-\epsilon_1)^{l+r-1}}. \tag{5.4.7}$$

Similarly choose $\epsilon_2 > 0$ so small that

$$\frac{1}{(\rho-\epsilon_2)^{l+r+1}} < \frac{1}{\rho^{l+r}} \qquad |z_j| > \rho$$

and choose $\eta_2 > 0$ such that

$$0 < \eta_2 < \frac{|z_j|}{\rho^{l+r}} - \frac{|z_j|}{(\rho-\epsilon_2)^{l+r+1}}. \tag{5.4.8}$$

Let

$$\epsilon = min(\epsilon_1, \epsilon_2) \qquad \text{and} \qquad \eta = min(\eta_1, \eta_2).$$

From (5.4.7) we have

$$\eta < \frac{|z_j|}{\rho^{l+r}} - \frac{1}{(\rho-\epsilon)^{l+r-1}}$$

or,

$$\frac{1}{(\rho-\epsilon)^{l+r-1}} < \frac{|z_j|}{\rho^{l+r}} - \eta$$

or,

$$\frac{1}{(\rho-\epsilon)^{(l+r-1)n}} < \left(\frac{|z_j|}{\rho^{l+r}} - \eta\right)^n. \tag{5.4.9}$$

Similarly from (5.4.8) we have

$$\eta < \frac{|z_j|}{\rho^{l+r}} - \frac{|z_j|}{(\rho-\epsilon)^{l+r+1}}$$

or,

$$\frac{|z_j|}{(\rho-\epsilon)^{(l+r+1)n}} < \left(\frac{|z_j|}{\rho^{l+r}} - \eta\right)^n. \tag{5.4.10}$$

From (5.4.6), (5.4.9) and (5.4.10) we have

$$\Delta_{rn-1,l}^{(p_j-1)}(z_j;f) = \mathcal{O}\left(\frac{|z_j|}{\rho^{l+r}} - \eta\right)^n \qquad \text{for} \qquad |z_j| > \rho. \tag{5.4.11}$$

Here and elsewhere η will denote sufficiently small positive number which is not same at each occurence. Now, let for any integer $n > 0$, p and q be determined by

$$(l+r-1)p + q = (l+r)n, \qquad 0 \le q < l+r-1.$$

Then for $0 \le j \le \mu$ from (5.4.4) we have

$$\begin{aligned}
\left(\sum_{k=0}^{n-1} a_{k+(r+l-1)n} z_j^k\right)^{(p_j-1)} &= \left(\sum_{k=(r+l-1)n}^{(r+l-1)n+n-1} a_k z_j^{k-n(r+l-1)}\right)^{(p_j-1)} \\
&= \left(\sum_{k=(r+l-1)n}^{p(r+l-1)-1} a_k z_j^{k-n(r+l-1)} + \right. \\
&\qquad \left. \sum_{k=p(r+l-1)}^{(l+r)n-1} a_k z_j^{k-n(r+l-1)}\right)^{(p_j-1)} \\
&= \left(\sum_{\nu=n}^{p-1} \sum_{k=0}^{(r+l-2)} a_{k+(r+l-1)\nu} z_j^{k+(r+l-1)(\nu-n)} + \right. \\
&\qquad \left. \sum_{k=p(r+l-1)}^{(l+r)n-1} a_k z_j^{k-(r+l-1)n}\right)^{(p_j-1)} \\
&= \left(\sum_{k=0}^{q-1} a_{k+p(r+l-1)} z_j^{k+(r+l-1)(p-n)}\right)^{(p_j-1)} \\
&= \mathcal{O}\left(\frac{|z_j|^{(r+l-1)(p-n)}}{(\rho-\epsilon)^{p(r+l-1)}}\right) \\
&= \mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{(l+r)n}}\right)
\end{aligned}$$

whence for $0 \leq j \leq \mu$ we have

$$\begin{aligned}
\Delta^{(p_j-1)}_{rn-1,1}(z_j;f) &= \left(\sum_{k=0}^{n-1} \beta_{l,r}(z_j^n) a_{k+(r+l-1)n} z_j^k + \right.\\
&\qquad \left. \sum_{q=l+1}^{\infty} \sum_{k=0}^{n-1} a_{k+(r+q-1)n} z_j^k \right)^{(p_j-1)} \\
&= \mathcal{O} max(1, |z_j|^{n(r-1)}) \left(\frac{|z_j|^n}{(\rho-\epsilon)^{(l+r)n}} + \frac{1}{(\rho-\epsilon)^{(r+l)n}} \right). \qquad (5.4.12)
\end{aligned}$$

Proceeding as before we can show that for $|z| < \rho$ by choosing ϵ sufficiently small we can find $\eta > 0$ such that

$$\frac{|z_j|^n}{(\rho-\epsilon)^{(r+l)n}} < \left(\frac{1}{\rho^{r+l-1}} - \eta \right)^n \qquad (5.4.13)$$

and

$$\frac{1}{(\rho-\epsilon)^{(l+r)n}} < \left(\frac{1}{\rho^{r+l-1}} - \eta \right)^n . \qquad (5.4.14)$$

From (5.4.12), (5.4.13) and (5.4.14) we have

$$\Delta^{(p_j-1)}_{rn-1,l}(z_j;f) = \mathcal{O}\left(max(1, |z_j|^{n(r-1)}) \left(\frac{1}{\rho^{r+l-1}} - \eta \right)^n \right) \qquad \text{for} \qquad |z_j| < \rho. \qquad (5.4.15)$$

Hence (5.4.11) and (5.4.15) gives

$$H^{(p_j-1)}_{l,r}(z_j;f) < K^1_{l,r}(|z_j|, \rho).$$

For the convers part suppose $H^{(p_j-1)}_{l,r}(z_j;f) < K^1_{l,r}(|z_j|,\rho)$ $(j = 1, 2, \ldots, s)$ for some $f = \sum_{k=0}^{\infty} a_k z^k \in A_\rho$ and that $rank\ M = (l+r-1)(l+r)$. Set

$$h(z) = \beta_{l,r}(z^{n+1}) \Delta_{rn-1,l}(z;f) - z^{r+l-1} \beta_{l,r}(z^n) \Delta_{r(n+1)-1,l}(z;f)$$

then from (4.12) [20] we have

$$\begin{aligned}
h(z) &= \beta_{l,r}(z^n)\beta_{l,r}(z^{n+1}) \left(\sum_{k=0}^{r+l-2} a_{k+(l+r-1)n} z^k - \sum_{k=0}^{r+l-1} a_{k+(l+r)n} z^{k+n} \right) \\
&\quad + \beta_{l,r}(z^{n+1}) \Delta_{rn-1,l+1}(z;f) - z^{r+l-1} \beta_{l,r}(z^n) \Delta_{r(n+1)-1,l+1}(z;f).
\end{aligned}$$

Thus, for $|z| \neq 1$ when $\beta_{l,r}(z^n)$ and $\beta_{l,r}(z^{n+1})$ has a zero on the unit circle then,

$$\begin{aligned}
H(z) &= \frac{h(z)}{\beta_{l,r}(z^n)\beta_{l,r}(z^{n+1})} \qquad (5.4.16) \\
&= \left(\sum_{k=0}^{r+l-2} a_{k+(l+r-1)n} z^k - \sum_{k=0}^{r+l-1} a_{k+(l+r)n} z^{k+n} \right) \\
&\quad + \frac{\Delta_{rn-1,l+1}(z;f)}{\beta_{l,r}(z^n)} - z^{r+l-1} \frac{\Delta_{r(n+1)-1,l+1}(z;f)}{\beta_{l,r}(z^{n+1})} \\
&= \left(\sum_{k=0}^{r+l-2} a_{k+(l+r-1)n} z^k - \sum_{k=0}^{r+l-1} a_{k+(l+r)n} z^{k+n} \right) \\
&\quad + \Gamma_{rn-1,l+1}(z;f) - z^{r+l-1} \Gamma_{r(n+1)-1,l+1}(z;f)
\end{aligned}$$

where

$$\Gamma_{rn-1,l+1}(z;f) = \frac{\Delta_{rn-1,l+1}(z;f)}{\beta_{l,r}(z^n)}.$$

Note that from Lemma 5.4.1 if $\beta_{l,r}(z^n)$ has no zero on the unit circle then,

$$\frac{1}{\beta_{l,r}(z^n)} = \mathcal{O}\begin{cases} 1 & |z| \leq 1, \\ \frac{1}{|z|^{n(r-1)}}. & 1 < |z| \end{cases} \tag{5.4.17}$$

Since from (5.1.7)

$$\Delta_{rn-1,l+1}(z;f) = \mathcal{O}\begin{cases} \frac{1}{\rho^{(r+l)n}} & |z| \leq 1, \\ \frac{|z|^{n(r-1)}}{\rho^{(r+l)n}} & 1 \leq |z| \leq \rho, \\ \frac{|z|^{n+n(r-1)}}{\rho^{(r+l+1)n}}. & \rho \leq |z| \end{cases}$$

Thus,

$$\Gamma_{rn-1,l+1}(z;f) = \mathcal{O}\begin{cases} 1/\rho^{(r+l)n} & |z| \leq 1, \\ 1/\rho^{(r+l)n} & 1 \leq |z| \leq \rho, \\ |z|^n/\rho^{(r+l+1)n} & \rho \leq |z|. \end{cases} \tag{5.4.18}$$

Let

$$K^2{}_{l,r}(R,\rho) = \begin{cases} 1/\rho^{1+(l-1)/r} & |z| \leq 1, \\ 1/\rho^{1+(l-1)/r} & 1 \leq |z| \leq \rho, \\ |z|^{1/r}/\rho^{1+l/r} & \rho \leq |z|. \end{cases}$$

Thus,

$$\begin{aligned} H^{(p_j-1)}(z) &= \left(\sum_{k=0}^{r+l-2} a_{k+(l+r-1)n} z^k - \sum_{k=0}^{r+l-1} a_{k+(l+r)n} z^{k+n}\right. \\ &\quad \left. +\Gamma_{rn-1,l+1}(z;f) z^{r+l-1} \Gamma_{r(n+1)-1,l+1}(z;f)\right)^{(p_j-1)} \\ &= \left(\sum_{k=0}^{r+l-2} a_{k+(l+r-1)n} z^k - \sum_{k=0}^{r+l-1} a_{k+(l+r)n} z^{k+n}\right)^{(p_j-1)} \\ &\quad +\mathcal{O}((K^2{}_{l+1,r}(|z|,\rho-\epsilon))^{rn}). \end{aligned} \tag{5.4.19}$$

For $0 \leq j \leq \mu$ from (5.4.13) and (5.4.14)

$$\begin{aligned} H^{(p_j-1)}(z_j) &= \left(\sum_{k=0}^{r+l-2} a_{k+(r+l-1)n} z_j^k\right)^{(p_j-1)} + \\ &\quad +\mathcal{O}\left(\frac{|z_j|^n}{(\rho-\epsilon)^{(l+r)n}} + \frac{1}{(\rho-\epsilon)^{(l+r)n}}\right) \\ &= \left(\sum_{k=0}^{l+r-2} a_{k+(r+l-1)n} z_j^k\right)^{(p_j-1)} + \mathcal{O}\left(\frac{1}{\rho^{r+l-1}} - \eta\right)^n. \end{aligned} \tag{5.4.20}$$

For any integer $t \geq 0$, let us denote

$$H^{1}{}^{t}_{l,r}(z;f) := \overline{\lim_{n\to\infty}}|\Gamma^{(t)}_{rn-1,l}(z;f)|^{1/rn}.$$

Now from the hypothesis $H^{(p_j-1)}_{l,r}(z_j;f) < K^1_{l,r}(|z_j|,\rho)$ $(j = 1,2,\ldots,\mu)$. This together with (5 4.16) and (5.4.17) give

$$H^{1}{}^{(p_j-1)}_{l,r}(z_j;f) < K^{2}{}_{l,r}(|z_j|,\rho). \qquad (j = 1,2,\ldots,\mu) \tag{5.4.21}$$

That is

$$\overline{\lim_{n\to\infty}}|\Gamma^{(p_j-1)}_{rn-1,l}(z;f)|^{1/rn} = K^{2}{}_{l,r}(|z_j|,\rho) - \eta$$

for some $\eta > 0$. Thus,

$$\Gamma^{(p_j-1)}_{rn-1,l}(z_j;f) \leq \left(K^{2}{}_{l,r}(|z_j|,\rho) - \eta + \epsilon\right)^{rn}$$

for $n \geq n_0(\epsilon)$ and $\eta > \epsilon > 0$. Thus,

$$\begin{aligned}
H^{(p_j-1)}(z_j) &= \Gamma^{(p_j-1)}_{rn-1,l}(z_j;f) - z_j^{r+l-1}\Gamma^{(p_j-1)}_{r(n+1)-1,l}(z_j;f) \\
&= \mathcal{O}\left(\frac{1}{\rho^{1+(l-1)/r}} - \eta\right)^{rn} \\
&= \mathcal{O}\left(\frac{1}{\rho^{r+l-1}} - \eta\right)^{n}.
\end{aligned}$$

Hence from (5.4.20) we obtain

$$\left(\sum_{k=0}^{r+l-2} a_{k+(r+l-1)n} z_j^k\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{r+l-1}} - \eta\right)^{n}. \tag{5.4.22}$$

Similarly for $j > \mu$ from (5.4.19) from (5.4.9) and (5.4.10) we have

$$\begin{aligned}
H^{(p_j-1)}(z_j) &= \left(-\sum_{k=0}^{r+l-1} a_{k+(l+r-1)n+n} z_j^{k+n}\right)^{(p_j-1)} + \\
&\quad + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(r+l-1)n}} + \frac{|z_j|^n}{(\rho-\epsilon)^{(r+l+1)n}}\right) \\
&= \left(-\sum_{k=0}^{r+l-1} a_{k+(l+r)n} z_j^{k+n}\right)^{(p_j-1)} + \mathcal{O}\left(\frac{|z_j|}{\rho^{r+l}} - \eta\right)^{n}.
\end{aligned} \tag{5.4.23}$$

Now from (5.4.21) $H^{1}{}^{(p_j-1)}_{l,r}(z_j;f) < K^{2}{}_{l,r}(|z_j|,\rho)$ $\quad (j = \mu+1,\ldots,s)$. That is

$$\overline{\lim_{n\to\infty}}|\Gamma^{(p_j-1)}_{rn-1,l}(z_j;f)|^{1/rn} = \frac{|z_j|^{1/r}}{\rho^{(1+l/r)}} - \eta$$

for some $\eta > 0$. Thus,

$$\Gamma^{(p_j-1)}_{rn-1,l}(z_j;f) \le \left(\frac{|z_j|^{1/r}}{\rho^{1+l/r}} - \eta + \epsilon\right)^{rn}$$

for $n \ge n_0(\epsilon)$ and $\eta > \epsilon > 0$. Thus

$$\begin{aligned} H^{(p_j-1)}(z_j) &= \Gamma^{(p_j-1)}_{rn-1,l}(z_j;f) - z_j^{r+l-1}\Gamma^{(p_j-1)}_{r(n+1)-1,l}(z_j;f) \\ &= \mathcal{O}\left(\frac{|z_j|^{1/r}}{\rho^{1+l/r}} - \eta\right)^{rn} \\ &= \mathcal{O}\left(\frac{|z_j|}{\rho^{r+l}} - \eta\right)^{n}. \end{aligned}$$

Hence from (5.4.23) we obtain

$$\left(\sum_{k=0}^{r+l-1} a_{k+(r+l)n} z_j^{k+n}\right)^{(p_j-1)} = \mathcal{O}\left(\frac{|z_j|}{\rho^{l+r}} - \eta\right)^{n}$$

or,

$$\left(\sum_{k=0}^{r+l-1} a_{k+(l+r)n} z_j^{k}\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{l+r}} - \eta_1\right)^{n}. \tag{5.4.24}$$

Now, since (5.4.22) and (5.4.24) hold for all n, putting $n = (l+r)\nu + \lambda, \lambda = 0, \ldots, l+r-1$ in (5.4.22) and $n = (l+r-1)\nu + \lambda, \lambda = 0, \ldots, l+r-2$ in (5.4.24) we have

$$\left(\sum_{k=0}^{l+r-2} a_{k+(l+r-1)(l+r)\nu+\lambda(l+r-1)} z_j^{k}\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{l+r-1}} - \eta\right)^{(l+r)\nu-\lambda} \tag{5.4.25}$$

$(j = 1, \ldots, \mu; \lambda = 0, 1, \ldots l+r-1; \nu = 0, 1, \ldots)$,

$$\left(\sum_{k=0}^{l+r-1} a_{k+(l+r)(l+r-1)\nu+\lambda(l+r)} z_j^{k}\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{l+r}} - \eta\right)^{(l+r-1)\nu+\lambda} \tag{5.4.26}$$

$(j = \mu+1, \ldots, s; \lambda = 0, 1, \ldots l+r-2; \nu = 0, 1, \ldots)$.

Now since

$$\frac{1}{\rho^{l+r-1}} - \eta < \frac{1}{\rho^{l+r-1}}, \qquad \eta > 0$$

$$\left(\frac{1}{\rho^{l+r-1}} - \eta\right)^{l+r} < \frac{1}{\rho^{(l+r-1)(l+r)}}$$

choose η_1 such that

$$0 < \eta_1 < \frac{1}{\rho^{(l+r-1)(l+r)}} - \left(\frac{1}{\rho^{l+r-1}} - \eta\right)^{l+r}$$

or,

$$\left(\frac{1}{\rho^{l+r-1}} - \eta\right)^{(l+r)\nu} < \left(\frac{1}{\rho^{(l+r-1)(l+r)}} - \eta_1\right)^{\nu}$$

hence (5.4.25) can be written as

$$\left(\sum_{k=0}^{l+r-2} a_{k+(l+r-1)(l+r)\nu+\lambda(l+r-1)} z_j^k\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{(l+r-1)(l+r)}} - \eta\right)^{\nu} \tag{5.4.27}$$

$(j = 1, \ldots, \mu; \lambda = 0, 1, \ldots l + r - 1; \nu = 0, 1, \ldots)$.

Similarly (5.4.26) can be written as

$$\left(\sum_{k=0}^{l+r-1} a_{k+(l+r)(l+r-1)\nu+\lambda(l+r)} z_j^k\right)^{(p_j-1)} = \mathcal{O}\left(\frac{1}{\rho^{(l+r-1)(l+r)}} - \eta\right)^{\nu} \tag{5.4.28}$$

$(j = \mu + 1, \ldots, s; \lambda = 0, 1, \ldots l + r - 2; \nu = 0, 1, \ldots)$.

Note that (5.4.27) and (5.4.28) can be written as

$$M.A^T = B \tag{5.4.29}$$

where

$$A = \left(a_{(l+r-1)(l+r)\nu}, a_{(l+r-1)(l+r)\nu+1}, \ldots, a_{(l+r-1)(l+r)\nu+(l+r-1)(l+r)-1}\right)$$

and

$$B = \left(\mathcal{O}\left(\frac{1}{\rho^{(l+r-1)(l+r)}} - \eta\right)^{\nu}\right),$$

B is a column vector of order $((s(l + r - 1) + \mu) \times 1)$.

Since $rank\ M = (l + r - 1)(l + r)$, solving (5.4.29) we get

$$a_{(l+r-1)(l+r)\nu+k} = \mathcal{O}\left(\frac{1}{\rho^{(l+r-1)(l+r)}} - \eta\right)^{\nu}$$

for $k = 0, 1, \ldots, (l + r - 1)(l + r) - 1$. Hence

$$\overline{\lim_{\nu\to\infty}} |a_\nu|^{1/\nu} < \frac{1}{\rho}$$

which is a contradiction to $f \in A_\rho$.

Remark 5.4.1 For $p_\nu = 1, \forall \nu$ Theorem 5.4.1 gives Theorem 5.1.2.

Remark 5.4.2 For $r = 1$ Theorem 5.4.1 gives Theoren 2.1.9.

Chapter 6

WALSH OVERCONVERGENCE USING POLYNOMIAL INTERPOLANTS IN Z AND Z^{-1}

6.1 Let $A_\rho (1 < \rho < \infty)$ denote the class of functions $f(z)$,analytic in $|z| < \rho$ and having a singularity on the circle $|z| = \rho$. For each ordered pair (m_i, n_i) of non-negative integers, and $q_0 > 0, q_0 > p \geq 0$ let $q_i = (m_i + n_i)q_0 + p$. For any $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in A_ρ, let $L_{q_i-1}(z; f)$ be the Lagrange interpolant of $z^{n_i} f(z)$ in Π_{q_i-1} at the $q_i{}^{th}$ roots of unity, then $A_{m_i+n_i-1}(z; f) = S_{m_i+n_i-1}(z; L_{q_i-1}(z; f)) = \sum_{j=0}^{m_i+n_i-1} \alpha_j z^j$ where $S_{n-1}(z : g)$ denotes the $(n-1)^{th}$ partial sum of the power series of $g(z)$. Thus $z^{-n_i} A_{(m_i+n_i-1)}(z; f)$ can be uniquely expressed as the sum of a polynomial in Π_{m_i-1} in the variable z and a polynomial in Π_{n_i} in the variable z^{-1}, that is

$$\begin{aligned}
z^{-n_i} A_{(m_i+n_i-1)}(z; f) &= z^{-n_i} \left(\sum_{j=0}^{n_i-1} \alpha_j z^j + \sum_{j=n_i}^{m_i+n_i-1} \alpha_j z^j \right) \\
&= \sum_{j=0}^{n_i-1} \alpha_j z^{j-n_i} + z^{-n_i} \sum_{j=0}^{{}_i-1} \alpha_{j+n_i} z^{j+n_i} \\
&= \sum_{j=0}^{n_i-1} \alpha_j z^{j-n_i} + \sum_{j=0}^{m_i-1} \alpha_{j+n_i} z^j \\
&= r_{n_i,m_i}^{q_0,p}(z^{-1}; f) + s_{m_i,n_i}^{q_0,p}(z; f). \qquad \text{(say)} \qquad (6.1.1)
\end{aligned}$$

Now define for each $j = 0, 1, \ldots$

$$P_{m_i,n_i,j}^{q_0,p}(z; f) = \sum_{k=0}^{m_i-1} a_{jq_i+k} z^k \qquad (6.1.2)$$

and for each $j = 1, 2, \ldots$

$$Q_{n_i,m_i,j}^{q_0,p}(z^{-1}; f) = \sum_{k=0}^{n_i-1} a_{jq_i-n_i+k} z^{k-n_i}. \qquad (6.1.3)$$

Finally for each positive integer l, we put

$$\Delta^{q_0,p}_{m_\imath,n_\imath,l}(z;f) = s^{q_0,p}_{m_\imath,n_\imath}(z;f) - \sum_{j=0}^{l-1} P^{q_0,p}_{m_\imath,n_\imath,j}(z;f), \tag{6.1.4}$$

$$\Theta^{q_0,p}_{n_\imath,m_\imath,l}(z^{-1};f) = r^{q_0,p}_{n_\imath,m_\imath}(z^{-1};f) - \sum_{j=1}^{l-1} Q^{q_0,p}_{n_\imath,m_\imath,j}(z^{-1};f). \tag{6.1.5}$$

With the above notations genralizing a result of Walsh [58,p.153] Cavaretta et al [12] established

Theorem 6.1.1 [12] *Let $f(z) \in A_\rho$ and $\{(m_\imath,n_\imath)\}_{\imath=1}^{\infty}$ be any sequence of ordered pairs of non-negative integers for which there exists an α with $0 \le \alpha < \infty$ such that*

$$\lim_{\imath\to\infty} m_\imath = \infty \qquad \textit{and} \qquad \lim_{\imath\to\infty} \frac{n_i}{m_\imath} = \alpha. \tag{6.1.6}$$

Then for each positive integer l

$$\lim_{\imath\to\infty} \left\{ s^{1,0}_{m_\imath,n_\imath}(z;f) - \sum_{j=0}^{l-1} P^{1,0}_{m_\imath,n_\imath,j}(z;f) \right\} = 0, \qquad \forall |z| < \rho^{1+l(1+\alpha)}$$

and for $\alpha > 0, l \ge 2$

$$\lim_{\imath\to\infty} \left\{ r^{1,0}_{n_\imath,m_\imath}(z^{-1};f) - \sum_{j=1}^{l-1} Q^{1,0}_{n_\imath,m_\imath,j}(z^{-1};f) \right\} = 0, \qquad \forall |z| > \rho^{1-l(1+\frac{1}{\alpha})}.$$

In this chapter, motivated by the results of Totik [56] and Sharma on l_2 approximation, we generalize and extend Theorem 6.1.1. In section 6.2 we give convergence theorems for polynomials in z and for polynomials in z^{-1} seperatly, and in section 6.3 we give their exact form. Further in section 6.4 we consider pointwise behaviour of $\Delta^{q_0,p}_{\imath,n_\imath,l}(z,f)$ and finally in section 6.5 we consider pointwise behaviour of $\Theta^{q_0,p}_{n_\imath,\imath,l}(z^{-1},f)$.

6.2 In this section we prove convergence theorem, generalizing Theorem 6.1.1, for the sequence $q_\imath = (m_\imath + n_\imath)q_0 + p, q_0 \ge 1, 0 \le p < q_0$ and $(m_\imath, n_\imath)$ satisfying (6.1.6), for polynomials in z and z^{-1}.

Let

$$h_{l,q_0,\alpha,p}(R) = \overline{\lim_{\imath\to\infty}} \max_{|z|=R} |\Delta^{q_0,p}_{m_\imath,n_\imath,l}(z;f)|^{1/m_\imath},$$

$$g_{l,q_0,\alpha,p}(R^{-1}) = \overline{\lim_{i\to\infty}} \max_{|z|=R} |\Theta^{q_0,p}_{n_\imath,m_\imath,l)}(z^{-1};f)|^{1/n_\imath}.$$

$$\begin{aligned} K_{l,q_0,\alpha}(R,\rho) &= \frac{R}{\rho^{1+l(1+\alpha)q_0}} \qquad \text{if} \qquad R \geq \rho \\ &= \frac{1}{\rho^{l(1+\alpha)q_0}} \qquad \text{if} \qquad 0 \leq R < \rho \end{aligned}$$

and for $\alpha > 0$

$$\begin{aligned} B_{l,q_0,\alpha}(R^{-1},\rho) &= \frac{1}{\rho^{l(1+\frac{1}{\alpha})q_0}} \qquad \text{if} \qquad R \geq \rho \\ &= \frac{\rho}{R\rho^{l(1+\frac{1}{\alpha})q_0}} \qquad \text{if} \qquad 0 < R < \rho. \end{aligned}$$

Theorem 6.2.1 *Let $f(z) \in A_\rho$ and $\{(m_i,n_i)\}_{i=1}^{\infty}$ be any sequence of ordered pairs of non-negative integers satisfying (6.1.6) then for each positive integer l*

$$h_{l,q_0,\alpha,p}(R) \leq K_{l,q_0,\alpha}(R,\rho). \tag{6.2.1}$$

Specifically

$$\lim_{i\to\infty}\left\{ s^{q_0,p}_{m_i,n_i}(z;f) - \sum_{j=0}^{l-1} P^{q_0,p}_{m_i,n_i,j}(z;f)\right\} = 0 \qquad \forall |z| < \rho^{1+lq_0(1+\alpha)}, \tag{6.2.2}$$

where the convergence is uniform and geometric for $|z| \leq Z < \rho^{1+lq_0(1+\alpha)}$. Moreover the result of (6.2.2) is best possible in the sense that (6.2.2) is not valid on $|z| = \rho^{1+lq_0(1+\alpha)}$ for all $f \in A_\rho$ and all sequences satisfying (6.1.6).

Before proving Theorem 6.2.1 we prove a lemma

Lemma 6.2.1 : For $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A_\rho$,

$$\Delta^{q_0,p}_{m_i,n_i,l}(z;f) = \sum_{k=0}^{m_i-1}\sum_{j=l}^{\infty} a_{jq_i+k} z^k$$

and

$$\Theta^{q_0,p}_{n_i,m_i,l}(z^{-1};f) = \sum_{k=0}^{n_i-1}\sum_{j=l}^{\infty} a_{jq_i-n_i+k} z^{k-n_i}.$$

Proof :

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k z^k \\ &= \sum_{k=0}^{q_i-n_i-1} a_k z^k + \sum_{k=q_i-n_i}^{\infty} a_k z^k \\ &= \sum_{k=0}^{q_i-n_i-1} a_k z^k + \sum_{k=0}^{q_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{jq_i-n_i+k}. \end{aligned}$$

So,

$$z^{n_i} f(z) = \sum_{k=0}^{q_i-n_i-1} a_k z^{k+n_i} + \sum_{k=0}^{q_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{jq_i+k}.$$

Now by the property

$$L_{q_i-1}(z; z^{aq_i} g(z)) = L_{q_i-1}(z; g(z))$$

and the fact that L_{q_i-1} reproduces polynomials of degree less than or equal to $q_i - 1$ we have

$$\begin{aligned} L_{q_i-1}(z^{n_i} f(z), z) &= L_{q_i-1}\Big(\sum_{k=0}^{q_i-n_i-1} a_k z^{k+n_i}\Big) + L_{q_i-1}\Big(\sum_{k=0}^{q_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{k+jq_i}\Big) \\ &= \sum_{k=0}^{q_i-n_i-1} a_k z^{k+n_i} + \sum_{j=1}^{\infty} L_{q_i-1}\Big(\sum_{k=0}^{q_i-1} a_{jq_i-n_i+k} z^{k+jq_i}\Big) \\ &= \sum_{k=0}^{q_i-n_i-1} a_k z^{k+n_i} + \sum_{j=1}^{\infty}\sum_{k=0}^{q_i-1} a_{jq_i-n_i+k} z^{k}. \end{aligned}$$

Thus,

$$A_{m_i+n_i-1}(z; f) = \sum_{k=0}^{m_i-1} a_k z^{k+n_i} + \sum_{k=0}^{m_i+n_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{k}$$

and hence

$$\begin{aligned} z^{-n_i} A_{(m_i+n_i-1)}(z; f) &= \sum_{k=0}^{m_i-1} a_k z^k + \sum_{k=0}^{m_i+n_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{k-n_i} \\ &= \sum_{k=0}^{m_i-1} a_k z^k + \sum_{k=0}^{n_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{k-n_i} + \\ &\quad \sum_{k=n_i}^{m_i+n_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{k-n_i} \\ &= \sum_{k=0}^{m_i-1} a_k z^k + \sum_{k=0}^{m_i-1}\sum_{j=1}^{\infty} a_{jq_i+k} z^k + \sum_{k=0}^{n_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i-k} z^{k-n_i} \\ &= \sum_{k=0}^{m_i-1}\sum_{j=0}^{\infty} a_{jq_i+k} z^k + \sum_{k=0}^{n_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{k-n_i}. \end{aligned} \tag{6.2.3}$$

From (6.1.1) and (6.2.3)

$$s^{q_0,p}_{m_i,n_i}(z; f) = \sum_{k=0}^{m_i-1}\sum_{j=0}^{\infty} a_{jq_i+k} z^k \tag{6.2.4}$$

and

$$r^{q_0,p}_{n_i,m_i}(z^{-1}; f) = \sum_{k=0}^{n_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k} z^{k-n_i}. \tag{6.2.5}$$

Thus,

$$\Delta^{q_0,p}_{m_i,n_i,l}(z; f) = s^{q_0,p}_{m_i,n_i}(z; f) - \sum_{j=0}^{l-1} P_{m_i,n_i,j}(z; f)$$

$$
\begin{aligned}
&= \sum_{k=0}^{m_i-1}\sum_{j=0}^{\infty} a_{jq_i+k}z^k - \sum_{k=0}^{m_i-1}\sum_{j=0}^{l-1} a_{jq_i+k}z^k \\
&= \sum_{k=0}^{m_i-1}\sum_{j=l}^{\infty} a_{jq_i+k}z^k.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\Theta^{q_0,p}_{n_i,m_i,l}(z^{-1};f) &= r^{q_0,p}_{n_i,m_i}(z^{-1};f) - \sum_{j=0}^{l-1} Q_{n_i,m_i,j}(z^{-1};f) \\
&= \sum_{k=0}^{n_i-1}\sum_{j=1}^{\infty} a_{jq_i-n_i+k}z^{k-n_i} - \sum_{k=0}^{n_i-1}\sum_{j=1}^{l-1} a_{jq_i-n_i+k}z^{k-n_i} \\
&= \sum_{k=0}^{n_i-1}\sum_{j=l}^{\infty} a_{jq_i-n_i+k}z^{k-n_i}.
\end{aligned}
$$

Proof of Theorem 6.2.1 : Since $f \in A_\rho$, thus $a_k = \mathcal{O}((\rho-\epsilon)^{-k})$ for every $\rho - 1 > \epsilon > 0$ and $k \geq k_0(\epsilon)$. Let R be fixed and $|z| = R$. Then by above lemma 6.2.1

$$
\begin{aligned}
\Delta^{q_0,p}_{m_i,n_i,l}(z;f) &= \sum_{k=0}^{m_i-1}\sum_{j=l}^{\infty} a_{jq_i+k}z^k \\
&= \mathcal{O}\left(\sum_{k=0}^{m_i-1}\sum_{j=l}^{\infty} \frac{|z|^k}{(\rho-\epsilon)^{jq_i+k}}\right) \\
&= \mathcal{O}\begin{cases} \frac{R^{m_i}}{(\rho-\epsilon)^{lq_i+m_i}} & \text{if} \quad R \geq \rho \\ \frac{1}{(\rho-\epsilon)^{lq_i}} & \text{if} \quad 0 < R < \rho \end{cases}
\end{aligned}
$$

hence

$$
|\Delta^{q_0,p}_{m_i,n_i,l}(z;f)| \leq K \begin{cases} \left(\frac{R}{(\rho-\epsilon)^{lq_0(\frac{n_i}{m_i}+1)+\frac{pl}{m_i}+1}}\right)^{m_i} & \text{if} \quad R \geq \rho \\ \left(\frac{1}{(\rho-\epsilon)^{lq_0(\frac{n_i}{m_i}+1)+\frac{pl}{m_i}}}\right)^{m_i} & \text{if} \quad 0 < R < \rho \end{cases}
$$

where K is a constant which need not be same at each occurence. Thus,

$$
\begin{aligned}
h_{l,q_0,\alpha,p}(R) &= \overline{\lim_{i\to\infty}} \max_{|z|=R} |\Delta^{q_0,p}_{m_i,n_i,l}(z;f)|^{1/m_i} \\
&\leq \begin{cases} \frac{R}{(\rho-\epsilon)^{l(1+\alpha)q_0+1}} & \text{if} \quad R \geq \rho \\ \frac{1}{(\rho-\epsilon)^{l(1+\alpha)q_0}} & \text{if} \quad 0 < R < \rho \end{cases}
\end{aligned}
$$

since ϵ is arbitrary, hence

$$
h_{l,q_0,\alpha,p}(R) \leq K_{l,q_0,\alpha}(R,\rho).
$$

To show that the result of (6.2.2) is best possible, choose $f_1(z) = (\rho - z)^{-1}$ in A_ρ, and let $\{m_i\}_{i=1}^{\infty}$ be any sequence of non-negative integers with $\lim_{i\to\infty} m_i = \infty$. For any real

$\alpha \geq 0$, set $n_i = [\alpha m_i]$, the integer part of αm_i, so that (6.1.6) is valid. Thus,

$$\begin{aligned}
\Delta^{q_0,p}_{m_i,n_i,l}(z;f_1) &= \sum_{k=0}^{m_i-1}\sum_{j=l}^{\infty} a_{jq_i+k}z^k \\
&= \sum_{k=0}^{m_i-1}\sum_{j=l}^{\infty} \frac{1}{\rho^{jq_i+k+1}}z^k \\
&= \sum_{k=0}^{m_i-1} \frac{z^k}{\rho^{k+1}}\sum_{j=l}^{\infty}\frac{1}{\rho^{jq_i}} \\
&= \frac{(\rho^{m_i}-z^{m_i})}{(\rho-z)\rho^{m_i+(l-1)q_i}(\rho^{q_i}-1)}.
\end{aligned}$$

For $|z| = \rho^{1+lq_0(1+\alpha)}$, the above expression yeilds

$$\lim_{i\to\infty}[min\{|\Delta^{q_0,p}_{m_i,n_i,l}(z;f_1)| : |z| = \rho^{1+lq_0(1+\alpha)}\}]$$

$$\begin{aligned}
&\geq \lim_{i\to\infty} \frac{(\rho^{(1+lq_0(1+\alpha))m_i}-\rho^{m_i})}{(\rho+\rho^{1+lq_0(1+\alpha)})\rho^{m_i+(l-1)((m_i+[\alpha m_i])q_0+p)}(\rho^{(m_i+[\alpha m_i])q_0+p}+1)} \\
&\geq \lim_{i\to\infty} \frac{\rho^{lq_0(1+\alpha)m_i}(1-\rho^{-m_i})}{(\rho+\rho^{1+lq_0(1+\alpha)})\rho^{l((m_i+[\alpha m_i])q_0+p)}(1+\rho^{-((m_i+[\alpha m_i])q_0+p)})} \\
&\geq \lim_{i\to\infty} \frac{\rho^{lq_0(\alpha m_i-[\alpha m_i])}(1-\rho^{-m_i})}{\rho^{pl}(\rho+\rho^{1+lq_0(1+\alpha)})(1+\rho^{-((m_i+[\alpha m_i])q_0+p)})} \\
&\geq \frac{K}{\rho^{pl}(\rho+\rho^{1+lq_0(1+\alpha)})} > 0,
\end{aligned}$$

where K is some constant, showing that (6.2.2) of Theorem 6.2.1 is not valid at any point of the circle $|z| = \rho^{1+lq_0(1+\alpha)}$ in this case.

Theorem 6.2.2 *Let $f(z) \in A_\rho$ and $\{(m_i,n_i)\}_{i=1}^{\infty}$ be any sequence of ordered pairs of non-negative integers satisfying (6.1.6) then for each positive integer $l \geq 2$ and for $\alpha > 0$,*

$$g_{l,q_0,\alpha,p}(R^{-1}) \leq B_{l,q_0,\alpha}(R^{-1},\rho). \tag{6.2.6}$$

Specifically

$$\lim_{i\to\infty}\left\{r^{q_0,p}_{n_i,m_i}(z^{-1};f) - \sum_{j=1}^{l-1} Q^{q_0,p}_{n_i,m_i,j}(z^{-1};f)\right\} = 0 \qquad \forall |z| > \rho^{1-lq_0(1+\frac{1}{\alpha})}, \tag{6.2.7}$$

where the convergence is uniform and geometric for $|z| \geq Z > \rho^{1-lq_0(1+\frac{1}{\alpha})}$. Moreover the result of (6.2.7) is best possible in the sense that (6.2.7) is not valid on $|z| = \rho^{1-lq_0(1+\frac{1}{\alpha})}$ for all $f \in A_\rho$ and all sequences satisfying (6.1.6).

Proof : Since $f \in A_\rho$ implies $a_k = \mathcal{O}((\rho-\epsilon)^{-k})$ for every $\rho - 1 > \epsilon > 0$ and

$k \geq k_0(\epsilon)$. Let R be fixed, $|z| = R$ and if $R < \rho$ then we assume $\epsilon > 0$ so small that $R < (\rho - \epsilon)$ be satisfied, as well. Then by above lemma 6.2.1

$$\begin{aligned}
\Theta^{q_0,p}_{n_\imath,m_\imath,l}(z^{-1}; f) &= \sum_{k=0}^{n_\imath-1} \sum_{j=l}^{\infty} a_{jq_\imath - n_\imath + k} z^{k-n_\imath} \\
&= \mathcal{O}\left(\sum_{k=0}^{n_\imath-1} \sum_{j=l}^{\infty} \frac{|z|^{k-n_\imath}}{(\rho-\epsilon)^{jq_\imath - n_\imath + k}} \right) \\
&= \mathcal{O}\begin{cases} \frac{1}{(\rho-\epsilon)^{lq_\imath}} & \text{if} \quad R \geq \rho \\ \frac{(\rho-\epsilon)^{n_\imath}}{R^{n_\imath}(\rho-\epsilon)^{lq_i}} & \text{if} \quad 0 < R < \rho \end{cases}
\end{aligned}$$

hence

$$|\Theta^{q_0,p}_{n_\imath,m_\imath,l}(z^{-1}; f)| \leq K \begin{cases} \left(\frac{1}{(\rho-\epsilon)^{lq_0(1+\frac{m_\imath}{n_\imath}) + \frac{pl}{n_\imath}}} \right)^{n_\imath} & \text{if} \quad R \geq \rho \\ \left(\frac{(\rho-\epsilon)}{R(\rho-\epsilon)^{lq_0(1+\frac{m_\imath}{n_\imath}) + \frac{pl}{n_\imath}}} \right)^{n_\imath} & \text{if} \quad 0 < R < \rho. \end{cases}$$

Thus,

$$\begin{aligned}
g_{l,q_0,\alpha,p}(R^{-1}) &= \overline{\lim_{\imath\to\infty}} \max_{|z|=R} |\Theta^{q_0,p}_{n_\imath,m_\imath,l}(z^{-1}; f)|^{1/n_\imath} \\
&\leq \begin{cases} \frac{1}{(\rho-\epsilon)^{lq_0(1+\frac{1}{\alpha})}} & \text{if} \quad R \geq \rho \\ \frac{R^{-1}}{(\rho-\epsilon)^{lq_0(1+\frac{1}{\alpha})-1}} & \text{if} \quad 0 < R < \rho \end{cases}
\end{aligned}$$

since ϵ is arbitrary, hence

$$g_{l,q_0,\alpha,p}(R^{-1}) \leq B_{l,q_0,\alpha}(R^{-1}, \rho)$$

To show that the result of (6.2.7) is best possible, choose $f_1(z) = (\rho - z)^{-1}$ in A_ρ, and let $\{m_\imath\}_{\imath=1}^{\infty}$ be any sequence of non-negative integers with $\lim_{\imath\to\infty} m_\imath = \infty$. For any real $\alpha \geq 1$, set $n_\imath = [\alpha m_\imath]$, the integer part of $\alpha m_\imath$, so that (6.1.6) is valid. Thus,

$$\begin{aligned}
\Theta^{q_0,p}_{n_\imath,m_\imath,l}(z^{-1}; f_1) &= \sum_{k=0}^{n_\imath-1} \sum_{j=l}^{\infty} a_{jq_\imath - n_\imath + k} z^{k-n_\imath} \\
&= \sum_{k=0}^{n_\imath-1} \sum_{j=l}^{\infty} \frac{1}{\rho^{jq_\imath - n_\imath + k + 1}} z^{k-n_\imath} \\
&= \left(\frac{z}{\rho}\right)^{-n_\imath} \cdot \frac{(\rho^{n_\imath} - z^{n_\imath})}{(\rho - z)\rho^{n_\imath + (l-1)q_\imath}(\rho^{q_\imath} - 1)}.
\end{aligned}$$

For $|z| = \rho^{1-lq_0(1+\frac{1}{\alpha})}$, the above expression yeilds

$$\lim_{\imath\to\infty} [min\{|\Theta^{q_0,p}_{n_\imath,m_\imath,l}(z; f_1)| : |z| = \rho^{1-lq_0(1+\frac{1}{\alpha})}\}]$$

$$\geq \lim_{i\to\infty} \left(\frac{\rho^{1-lq_0(1+\frac{1}{\alpha})}}{\rho}\right)^{-n_\imath} \frac{(\rho^{n_\imath} - \rho^{(1-lq_0(1+\frac{1}{\alpha}))n_\imath})}{(\rho + \rho^{1-lq_0(1+\frac{1}{\alpha})})\rho^{n_\imath + (l-1)((m_\imath + [\alpha m_\imath])q_0 + p)}(\rho^{(m_\imath + [\alpha m_\imath])q_0 + p} + 1)}$$

$$\geq \lim_{\imath\to\infty} \frac{\rho^{lq_0(1+\frac{1}{\alpha})n_\imath}(1-\rho^{-lq_0(1+\frac{1}{\alpha})n_\imath})}{(\rho+\rho^{1-lq_0(1+\frac{1}{\alpha})})\rho^{l((m_\imath+n_\imath)q_0+p)}(1+\rho^{-((m_\imath+n_\imath)q_0+p)})}$$

$$\geq \lim_{\imath\to\infty} \frac{\rho^{lq_0(\frac{1}{\alpha}[\alpha m_\imath]-m_\imath)}(1-\rho^{-lq_0(1+\frac{1}{\alpha})n_\imath})}{(\rho+\rho^{1-lq_0(1+\frac{1}{\alpha})})\rho^{pl}(1+\rho^{-((m_\imath+[\alpha m_\imath])q_0+p)})}$$

$$\geq \frac{K}{\rho^{pl}(\rho+\rho^{1-lq_0(1+\frac{1}{\alpha})})} > 0,$$

showing that (6.2.7) of Theorem 6.2.2 is not valid at any point of the circle $|z| = \rho^{1-lq_0(1+\frac{1}{\alpha})}$ in this case.

Remark 6.2.1 For $p = 0, q_0 = 1$ Theorem 6.2.1 and Theorem 6.2.2 give Theorem 6.1.1.

Note that for $n_\imath = 0$, we have

$$s^{q_0,p}_{m_\imath,0}(z;f) = A_{m_\imath-1}(z;f) = S_{m_\imath-1}(z;L_{m_\imath q_0+p-1}(z;f))$$

which is the least square approximating polynomial of degree $m_\imath - 1$ to $f(z)$ at $q_\imath^{th}$ roots of unity where $q_\imath = m_\imath q_0 + p$.

Also

$$P^{q_0,p}_{m_\imath,0,\jmath}(z;f) = \sum_{k=0}^{m_\imath-1} a_{k+q_\imath\jmath} z^k.$$

Thus,

Remark 6.2.2 For $n_\imath = 0$ Theorem 6.2.1 give Theorem 5 [20] for sequence $m_\imath$.

Further, for $n_\imath = 0$ if $p = 0$ and $q_0 = 1$ then

$$s^{1,0}_{m_\imath,0}(z;f) = A_{m_\imath-1}(z;f) = S_{m_\imath-1}(z;L_{m_\imath-1}(z;f)) = L_{m_\imath-1}(z;f)$$

and

$$P^{1,0}_{m_\imath,0,\jmath}(z;f) = \sum_{k=0}^{m_\imath-1} a_{k+m_\imath\jmath} z^k.$$

Thus,

Remark 6.2.3 For $n_\imath = 0$ and $p = 0, q_0 = 1$ Theorem 6.2.1 give Theorem 1 [12] for sequence $m_\imath$.

6.3 In this section we extend Theorem (6.2.1) and Theorem (6.2.2) to show that equality holds in (6.2.1) and (6.2.6) for a special sequence. As a particular case we get extension of Theorem 6.1.1.

Let

$$m_i = i, \forall i \qquad \text{and} \qquad n_i = \alpha i + c, \ \forall i \tag{6.3.1}$$

with $0 \leq c < \alpha$, α is an integer ≥ 0. Thus $q_i = (m_i + n_i)q_0 + p = (1+\alpha)q_0 i + (p + q_0 c)$. Our result is that for this sequence $\{(i, n_i)\}$, equality holds in Theorem 6.2.1 and Theorem 6.2.2.

Theorem 6.3.1 *If $f \in A_\rho, \rho > 1, l$ is a positive integer and $R > 0$ then*

$$h_{l,q_0,\alpha,p}(R) = K_{l,q_0,\alpha}(R, \rho)$$

Proof : From Theorem 6.2.1 we have

$$h_{l,q_0,\alpha,p}(R) \leq K_{l,q_0,\alpha}(R, \rho).$$

To prove the opposite inequality first let us consider $R \geq \rho$, so

$$\begin{aligned}
\Delta^{q_0,p}_{i,n_i,l}(z; f) &= \sum_{k=0}^{i-1} \sum_{j=l}^{\infty} a_{jq_i+k} z^k \\
&= \sum_{k=0}^{i-1} a_{lq_i+k} z^k + \sum_{k=0}^{i-1} \sum_{j=l+1}^{\infty} a_{jq_i+k} z^k \\
&= \sum_{k=0}^{i-l(1+\alpha)q_0-2} a_{lq_i+k} z^k + \sum_{k=i-l(1+\alpha)q_0-1}^{i-1} a_{lq_i+k} z^k \\
&\quad + \sum_{k=0}^{i-1} \sum_{j=l+1}^{\infty} a_{jq_i+k} z^k \\
\sum_{k=i-l(1+\alpha)q_0-1}^{i-1} a_{lq_i+k} z^k &= \Delta^{q_0,p}_{i,n_i,l}(z; f) - \sum_{k=0}^{i-l(1+\alpha)q_0-2} a_{lq_i+k} z^k \\
&\quad - \sum_{k=0}^{i-1} \sum_{j=l+1}^{\infty} a_{jq_i+k} z^k.
\end{aligned}$$

Then by cauchy integral formula, for $i - l(1+\alpha)q_0 - 1 \leq k \leq i - 1$

$$\begin{aligned}
a_{lq_i+k} &= \frac{1}{2\pi i} \int_{|z|=R} \frac{\Delta^{q_0,p}_{i,n_i,l}(z; f)}{z^{k+1}} dz - \sum_{k'=0}^{i-l(1+\alpha)q_0-2} a_{lq_i+k'} \frac{1}{2\pi i} \int_{|z|=R} \frac{z^{k'}}{z^{k+1}} dz \\
&\quad - \frac{1}{2\pi i} \int_{|z|=R} \frac{\sum_{k'=0}^{i-1} \sum_{j=l+1}^{\infty} a_{jq_i+k'} z^{k'}}{z^{k+1}} dz \\
&= \frac{1}{2\pi i} \int_{|z|=R} \frac{\Delta^{q_0,p}_{i,n_i,l}(z; f)}{z^{k+1}} dz - \frac{1}{2\pi i} \int_{|z|=R} \frac{\sum_{k'=0}^{i-1} \sum_{j=l+1}^{\infty} a_{jq_i+k'} z^{k'}}{z^{k+1}} dz.
\end{aligned}$$

Hence by the definition of $h_{l,q_0,\alpha,p}(R)$ for every $\epsilon > 0$

$$\begin{aligned}
|a_{lq_\imath+k}| &\leq \frac{(h_{l,q_0,\alpha,p}(R)+\epsilon)^i}{R^k} + \mathcal{O}\left(\frac{R^\imath}{(\rho-\epsilon)^{(l+1)q_\imath+\imath}R^k}\right) \\
&\leq \frac{(h_{l,q_0,\alpha,p}(R)+\epsilon)^\imath}{R^k} + \mathcal{O}\left(\frac{R^{\imath-k}}{(\rho-\epsilon)^{(l+1)q_\imath+\imath}}\right) \\
&\leq \frac{(h_{l,q_0,\alpha,p}(R)+\epsilon)^\imath}{R^k} + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(l+1)(\imath+n_\imath)q_0+\imath}}\right)
\end{aligned}$$

now choose ϵ so small that

$$\frac{1}{(\rho-\epsilon)^{(l+1)(1+\alpha)q_0+1}} < \frac{1}{\rho^{l(1+\alpha)q_0+1}}.$$

Thus

$$\begin{aligned}
|a_{lq_\imath+k}| &\leq \frac{(h_{l,q_0,\alpha,p}(R)+\epsilon)^\imath}{R^k} + \mathcal{O}\left(\frac{1}{\rho^{l(\imath+n_\imath)q_0+\imath}}\right) \\
(h_{l,q_0,\alpha,p}(R)+\epsilon)^\imath &\geq R^k\left(|a_{lq_\imath+k}| - \mathcal{O}\left(\frac{1}{\rho l(i+n_\imath)q_0+i}\right)\right)
\end{aligned}$$

hence

$$h_{l,q_0,\alpha,p}(R)+\epsilon \geq \varlimsup_{\imath\to\infty} R^{\frac{k}{\imath}}\left(|a_{lq_\imath+k}|^{\frac{1}{lq_\imath+k}}\right)^{\frac{lq_\imath+k}{\imath}}.$$

Now since $i - l(1+\alpha)q_0 - 1 \leq k \leq i-1$ hence $\lim_{\imath\to\infty}\frac{k}{\imath} = 1$ thus,

$$h_{l,q_0,\alpha,p}(R)+\epsilon \geq \frac{R}{\rho^{l(1+\alpha)q_0+1}}$$

$\epsilon > 0$ being arbitrary, hence

$$h_{l,q_0,\alpha,p}(R) \geq \frac{R}{\rho^{1+l(1+\alpha)q_0}} \qquad \text{for } R \geq \rho.$$

For $0 < R < \rho$ in the same manner we have

$$\begin{aligned}
\Delta^{q_0,p}_{\imath,n_\imath,l}(z;f) &= \sum_{k=0}^{\imath-1}\sum_{\jmath=l}^{\infty} a_{\jmath q_\imath+k}z^k \\
&= \sum_{k=0}^{\imath-1} a_{lq_\imath+k}z^k + \sum_{k=0}^{\imath-1}\sum_{\jmath=l+1}^{\infty} a_{\jmath q_\imath+k}z^k \\
&= \sum_{k=0}^{l(1+\alpha)q_0-1} a_{lq_\imath+k}z^k + \sum_{k=l(1+\alpha)q_0}^{\imath-1} a_{lq_\imath+k}z^k \\
&\quad + \sum_{k=0}^{\imath-1}\sum_{\jmath=l+1}^{\infty} a_{\jmath q_\imath+k}z^k \\
\sum_{k=0}^{l(1+\alpha)q_0-1} a_{lq_\imath+k}z^k &= \Delta^{q_0,p}_{\imath,n_\imath,l}(z;f) - \sum_{k=l(1+\alpha)q_0}^{\imath-1} a_{lq_\imath+k}z^k \\
&\quad - \sum_{k=0}^{\imath-1}\sum_{\jmath=l+1}^{\infty} a_{\jmath q_\imath+k}z^k.
\end{aligned}$$

Then by cauchy integral formula, for $0 \le k \le l(1+\alpha)q_0 - 1$ we have

$$\begin{aligned} a_{lq_\imath+k} &= \frac{1}{2\pi i}\int_{|z|=R} \frac{\Delta^{q_0,p}_{\imath,n_\imath,l}(z;f)}{z^{k+1}}dz - \sum_{k'=l(1+\alpha)q_0}^{\imath-1} a_{lq_\imath+k'}\frac{1}{2\pi i}\int_{|z|=R}\frac{z^{k'}}{z^{k+1}}dz \\ &\quad -\frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{k'=0}^{\imath-1}\sum_{j=l+1}^{\infty} a_{jq_\imath+k'}z^{k'}}{z^{k+1}}dz \\ &= \frac{1}{2\pi i}\int_{|z|=R} \frac{\Delta^{q_0,p}_{\imath,n_\imath,l}(z;f)}{z^{k+1}}dz - \frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{k'=0}^{\imath-1}\sum_{j=l+1}^{\infty} a_{jq_\imath+k'}z^{k'}}{z^{k+1}}dz. \end{aligned}$$

Hence by the definition of $h_{l,q_0,\alpha,p}(R)$ for every $\epsilon > 0$

$$\begin{aligned} |a_{lq_\imath+k}| &\le \frac{(h_{l,q_0,\alpha,p}(R)+\epsilon)^\imath}{R^k} + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(l+1)q_\imath}R^k}\right) \\ &\le \frac{(h_{l,q_0,\alpha,p}(R)+\epsilon)^\imath}{R^k} + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(l+1)q_\imath}}\right) \qquad (\text{since } 0 \le k). \end{aligned}$$

Choose ϵ so small that

$$\frac{1}{(\rho-\epsilon)^{(l+1)}} < \frac{1}{\rho^l}.$$

Thus,

$$(h_{l,q_0,\alpha,p}(R)+\epsilon)^\imath \ge R^k\left(|a_{lq_\imath+k}| - \mathcal{O}\left(\frac{1}{\rho^{lq_\imath}}\right)\right)$$

hence,

$$h_{l,q_0,\alpha,p}(R)+\epsilon \ge \overline{\lim_{\imath\to\infty}} R^{\frac{k}{\imath}}\left(|a_{lq_\imath+k}|^{\frac{1}{lq_\imath+k}}\right)^{\frac{lq_\imath+k}{\imath}}.$$

Now since $0 \le k \le l(1+\alpha)q_0 - 1$ hence $\lim_{\imath\to\infty}\frac{k}{\imath} = 0$ thus,

$$h_{l,q_0,\alpha,p}(R)+\epsilon \ge \frac{1}{\rho^{l(1+\alpha)q_0}}$$

$\epsilon > 0$ being arbitrary, hence

$$h_{l,q_0,\alpha,p}(R) \ge \frac{1}{\rho^{l(1+\alpha)q_0}} \qquad \text{for } 0 < R < \rho.$$

Thus

$$h_{l,q_0,\alpha,p}(R) \ge K_{l,q_0,\alpha}(R,\rho)$$

and the proof for polynomials in z is complete.

Theorem 6.3.2 *If $f \in A_\rho, \rho > 1, l \ge 2$ is a positive integer and $R > 0$ then for* $\alpha > 0$

$$g_{l,q_0,\alpha,p}(R^{-1}) = B_{l,q_0,\alpha}(R^{-1},\rho).$$

Proof : From Theorem 6.2.2 we have

$$g_{l,q_0,\alpha,p}(R^{-1}) \leq B_{l,q_0,\alpha}(R^{-1},\rho).$$

For the opposite inequlity first consider $R \geq \rho$, so

$$\begin{aligned}
\Theta^{q_0,p}_{n_\imath,\imath,l}(z^{-1};f) &= \sum_{k=0}^{n_\imath-1}\sum_{j=l}^{\infty} a_{jq_\imath-n_\imath+k}z^{k-n_\imath} \\
&= \sum_{k=0}^{n_\imath-1} a_{lq_\imath-n_\imath+k}z^{k-n_\imath} + \sum_{k=0}^{n_\imath-1}\sum_{j=l+1}^{\infty} a_{jq_\imath-n_\imath+k}z^{k-n_\imath} \\
&= \sum_{k=0}^{n_\imath-l(1+\alpha)q_0-1} a_{lq_\imath-n_\imath+k}z^{k-n_\imath} + \sum_{k=n_\imath-l(1+\alpha)q_0}^{n_\imath-1} a_{lq_\imath-n_\imath+k}z^{k-n_\imath} \\
&\quad + \sum_{k=0}^{n_\imath-1}\sum_{j=l+1}^{\infty} a_{jq_\imath-n_\imath+k}z^{k-n_\imath} \\
\sum_{k=n_\imath-l(1+\alpha)q_0}^{n_\imath-1} a_{lq_\imath-n_\imath+k}z^{k-n_\imath} &= \Theta^{q_0,p}_{n_\imath,\imath,l}(z^{-1};f) - \sum_{k=0}^{n_\imath-l(1+\alpha)q_0-1} a_{lq_\imath-n_\imath+k}z^{k-n_\imath} \\
&\quad -\sum_{k=0}^{n_\imath-1}\sum_{j=l+1}^{\infty} a_{jq_\imath-n_\imath+k}z^{k-n_\imath}.
\end{aligned}$$

Then by cauchy integral formula, for $n_\imath - l(1+\alpha)q_0 \leq k \leq n_\imath - 1$ we have

$$\begin{aligned}
a_{lq_\imath-n_\imath+k} &= \frac{1}{2\pi i}\int_{|z|=R}\frac{\Theta^{q_0,p}_{n_\imath,\imath,l}(z^{-1};f)}{z^{k+1-n_\imath}}dz - \\
&\quad - \sum_{k'=0}^{n_\imath-l(1+\alpha)q_0-1} a_{lq_\imath-n_\imath+k'}\frac{1}{2\pi i}\int_{|z|=R}\frac{z^{k'-n_\imath}}{z^{k+1-n_\imath}}dz \\
&\quad -\frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{k'=0}^{n_\imath-1}\sum_{j=l+1}^{\infty} a_{jq_\imath-n_\imath+k'}z^{k'-n_\imath}}{z^{k+1-n_\imath}}dz \\
&= \frac{1}{2\pi i}\int_{|z|=R}\frac{\Theta^{q_0,p}_{n_\imath,\imath,l}(z^{-1};f)}{z^{k+1-n_\imath}}dz - \\
&\quad -\frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{k'=0}^{n_\imath-1}\sum_{j=l+1}^{\infty} a_{jq_\imath-n_\imath+k'}z^{k'-n_\imath}}{z^{k+1-n_\imath}}dz.
\end{aligned}$$

Hence by the definition of $G_l(R^{-1})$ for every $\epsilon > 0$ and for $n_\imath - l(1+\alpha)q_0 \leq k \leq n_\imath - 1$

$$\begin{aligned}
|a_{lq_\imath-n_\imath+k}| &\leq \frac{(g_{l,q_0\,p}(R^{-1})+\epsilon)^{n_\imath}}{R^{k-n_\imath}} + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(l+1)q_\imath}}\right) \\
&\leq \frac{(g_{l,q_0,\alpha,p}(R^{-1})+\epsilon)^{n_\imath}}{R^{k-n_\imath}} + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(l+1)q_\imath}}\right).
\end{aligned}$$

Choose ϵ so small that

$$\frac{1}{\rho^{l(1+\frac{1}{\alpha})q_0-1}} > \frac{1}{(\rho-\epsilon)^{(l+1)(1+\frac{1}{\alpha})q_0}}$$

hence,

$$(g_{l,q_0,\alpha,p}(R^{-1})+\epsilon)^{n_i} \geq R^{k-n_i}\left(|a_{lq_i-n_i+k}| - \mathcal{O}\left(\frac{1}{\rho^{lq_i-n_i}}\right)\right).$$

Thus,

$$g_{l,q_0,\alpha,p}(R^{-1})+\epsilon \geq \overline{\lim_{i\to\infty}} R^{\frac{k-n_i}{n_i}}\left(|a_{lq_i-n_i+k}|^{\frac{1}{lq_i-n_i+k}}\right)^{\frac{lq_i-n_i+k}{n_i}}.$$

Now since $n_i - l(1+\alpha) \leq k \leq n_i - 1$ hence $\lim_{i\to\infty}\frac{k}{n_i} = 1$ thus,

$$g_{l,q_0,\alpha,p}(R^{-1})+\epsilon \geq \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}}$$

$\epsilon > 0$ being arbitrary, hence

$$g_{l,q_0,\alpha,p}(R^{-1}) \geq \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} \qquad \text{for } R \geq \rho.$$

For $0 < R < \rho$ in the same manner we have

$$\begin{aligned}
\Theta^{q_0,p}_{n_i,i,l}(z^{-1};f) &= \sum_{k=0}^{n_i-1}\sum_{j=l}^{\infty} a_{jq_i-n_i+k}z^{k-n_i} \\
&= \sum_{k=0}^{n_i-1} a_{lq_i-n_i+k}z^{k-n_i} + \sum_{k=0}^{n_i-1}\sum_{j=l+1}^{\infty} a_{jq_i-n_i+k}z^{k-n_i} \\
&= \sum_{k=0}^{l(1+\alpha)q_0-\alpha-1} a_{lq_i-n_i+k}z^{k-n_i} + \sum_{k=l(1+\alpha)q_0-\alpha}^{n_i-1} a_{lq_i-n_i+k}z^{k-n_i} \\
&\quad + \sum_{k=0}^{n_i-1}\sum_{j=l+1}^{\infty} a_{jq_i-n_i+k}z^{k-n_i} \\
\sum_{k=0}^{l(1+\alpha)q_0-\alpha-1} a_{lq_i-n_i+k}z^{k-n_i} &= \Theta^{q_0,p}_{n_i,i,l}(z^{-1};f) - \sum_{k=l(1+\alpha)q_0-\alpha}^{n_i-1} a_{lq_i-n_i+k}z^{k-n_i} \\
&\quad - \sum_{k=0}^{n_i-1}\sum_{j=l+1}^{\infty} a_{jq_i-n_i+k}z^{k-n_i}.
\end{aligned}$$

Then by cauchy integral formula, for $0 \leq k \leq l(1+\alpha)q_0 - \alpha - 1$ we have

$$\begin{aligned}
a_{lq_i-n_i+k} &= \frac{1}{2\pi i}\int_{|z|=R}\frac{\Theta^{q_0,p}_{n_i,i,l}(z^{-1};f)}{z^{k+1-n_i}}dz - \\
&\quad - \sum_{k'=l(1+\alpha)q_0-\alpha}^{n_i-1} a_{lq_i-n_i+k'}\frac{1}{2\pi i}\int_{|z|=R}\frac{z^{k-n_i'}}{z^{k+1-n_i}}dz \\
&\quad - \frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{k'=0}^{n_i-1}\sum_{j=l+1}^{\infty} a_{jq_i-n_i+k'}z^{k'-n_i}}{z^{k+1-n_i}}dz \\
&= \frac{1}{2\pi i}\int_{|z|=R}\frac{\Theta^{q_0,p}_{n_i,i,l}(z^{-1};f)}{z^{k+1-n_i}}dz - \\
&\quad - \frac{1}{2\pi i}\int_{|z|=R}\frac{\sum_{k'=0}^{n_i-1}\sum_{j=l+1}^{\infty} a_{jq_i-n_i+k'}z^{k'-n_i}}{z^{k+1-n_i}}dz.
\end{aligned}$$

Hence by the definition of $g_{l,q_0,\alpha,p}(R^{-1})$, since $0 \le k \le l(1+\alpha)q_0 - \alpha$ for every $\epsilon > 0$ we have

$$\begin{aligned} |a_{lq_i-n_i+k}| &\le \frac{(g_{l,q_0,\alpha,p}(R^{-1})+\epsilon)^{n_i}}{R^{k-n_i}} + \mathcal{O}\left(\frac{R^{-n_i}}{(\rho-\epsilon)^{(l+1)q_i-n_i}R^{k-n_i}}\right) \\ &\le \frac{(g_{l,q_0,\alpha,p}(R^{-1})+\epsilon)^{n_i}}{R^{k-n_i}} + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(l+1)q_i-n_i}}\right). \end{aligned}$$

Now choose ϵ so small that

$$\frac{1}{(\rho-\epsilon)^{(l+1)(1+\frac{1}{\alpha})q_0-1}} < \frac{1}{\rho^{l(1+\frac{1}{\alpha})q_0-1}}.$$

Hence,

$$(g_{l,q_0,\alpha,p}(R^{-1})+\epsilon)^{n_i} \ge R^{k-n_i}\left(|a_{lq_i-n_i+k}| - \mathcal{O}\left(\frac{1}{\rho^{lq_i-n_i}}\right)\right).$$

Thus,

$$g_{l,q_0,\alpha,p}(R^{-1})+\epsilon \ge \overline{\lim_{i\to\infty}} R^{\frac{k-n_i}{n_i}}\left(|a_{lq_i-n_i+k}|^{\frac{1}{lq_i-n_i+k}}\right)^{\frac{lq_i-n_i+k}{n_i}}.$$

Since $0 \le k \le l(1+\alpha)q_0 - \alpha$ hence $\lim_{i\to\infty}\frac{k}{n_i} = 0$ thus,

$$g_{l,q_0,\alpha,p}(R^{-1})+\epsilon \ge \frac{R^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} = \frac{\rho}{R\rho^{lq_0(1+\frac{1}{\alpha})}}$$

$\epsilon > 0$ being arbitrary, hence

$$g_{l,q_0,\alpha,p}(R^{-1}) \ge \frac{\rho}{R\rho^{lq_0(1+\frac{1}{\alpha})}} \qquad \text{for } 0 < R < \rho.$$

Thus,

$$g_{l,q_0,\alpha,p}(R^{-1}) \ge B_{l,q_0,\alpha}(R^{-1},\rho)$$

and the proof is complete.

Corollary 6.3.1 If $l \ge 1$, f is analytic in an open domain containing $|z| \le 1$ and $h_{l,q_0,\alpha,p}(R) = K_{l,q_0,\alpha}(R,\rho)$ or $g_{l,q_0,\alpha,p}(R^{-1}) = B_{l,q_0,\alpha}(R^{-1},\rho)$ for some $R > 0, \rho > 1$ then $f \in A_\rho$.

Proof : Given f is analytic in an open domain containing $|z| \le 1$.
Let $f \in A_{\rho'}, \rho' > 1$ then by Theorem 6.3.1 $h_{l,q_0,\alpha,p}(R) = K_{l,q_0,\alpha}(R,\rho')$, and by the hypothesis $h_{l,q_0,\alpha,p}(R) = K_{l,q_0,\alpha}(R,\rho)$ thus

$$K_{l,q_0,\alpha}(R,\rho') = K_{l,q_0,\alpha}(R,\rho)$$

which by the definition of $K_{l,q_0,\alpha}(R,\rho)$ gives $\rho = \rho'$, whence $f \in A_\rho$. Similar arguments can be given for the other case.

Remark 6.3.1 For $p = 0, q_0 = 1$ Theorem (6.3.1) extend Theorem 6.1.1.

6.4 We now study properties of a set containing points in $|z| < \rho$ and $|z| > \rho$ simultaneously, regarding the number of points in $|z| < \rho$ and $|z| > \rho$, by defining the concept of distinguished points for the polynomials in z.
sets

If we set

$$H_{l,q_0,\alpha}(z;f) = \overline{\lim_{i\to\infty}} |\Delta^{q_0,p}_{i,n_i,l}(z;f)|^{1/i}, \tag{6.4.1}$$

then from the result of Theorem 6.3.1 and the definition of $K_{l,q_0,\alpha}(|z|,\rho)$ it follows that $H_{l,q_0,\alpha}(z;f) \le K_{l,q_0,\alpha}(|z|,\rho)$. Set

$$\delta_{l,q_0,\alpha,\rho}(f) = \{z | H_{l,q_0,\alpha}(z;f) < K_{l,q_0,\alpha}(|z|,\rho)\}, \qquad f \in A_\rho, \qquad \rho > 1.$$

Define a set Z of points to be (l, q_0, α, ρ) distinguished if there is an $f \in A_\rho$ such that

$$H_{l,q_0,\alpha}(z_j;f) < K_{l,q_0,\alpha}(|z_j|,\rho),$$

for each $z_j \in Z$. That is $Z \subset \delta_{l,q_0,\alpha,\rho}(f)$. Suppose $Z = \{z_j\}_1^s$ is given in which $|z_j| < \rho$ $(j = 1,\ldots,\mu)$ and $|z_j| > \rho$ $(j = \mu+1,\ldots,s)$. We want to find a criterion to determine whether Z is (l, q_0, α, ρ) distinguished or not. Set the matrices X, Y, M (X, Y) as

$$X = \begin{pmatrix} 1 & z_1 & \ldots & z_1^{lq_0(1+\alpha)-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & z_\mu & \ldots & z_\mu^{lq_0(1+\alpha)-1} \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & z_{\mu+1} & \ldots & z_{\mu+1}^{lq_0(1+\alpha)} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & z_s & \ldots & z_s^{lq_0(1+\alpha)} \end{pmatrix}.$$

The matrices X and Y are of order $(\mu \times lq_0(1+\alpha))$ and $(s-\mu)\times(lq_0(1+\alpha)+1)$ respectively. Define

$$M = M(X,Y) = \begin{pmatrix} X & & & \\ & X & & 0 \\ & & \ddots & \\ & 0 & & X \\ Y & & & \\ & Y & & 0 \\ & & \ddots & \\ & 0 & & Y \end{pmatrix},$$

where X occurs $lq_0(1+\alpha)+1$ times and Y occurs $lq_0(1+\alpha)$ times beginning under the last X. The matrix M is of order $(slq_0(1+\alpha)+\mu) \times lq_0(1+\alpha)(lq_0(1+\alpha)+1)$. We now formulate

Theorem 6.4.1 *Suppose $Z = \{z_j\}_1^s$ is a set of points in C such that $|z_j| < \rho$ ($j = 1, \ldots, \mu$) and $|z_j| > \rho$ ($j = \mu+1, \ldots, s$). Then the set Z is (l, q_0, α, ρ) distinguished iff*

$$rank\ \ M < lq_0(1+\alpha)(lq_0(1+\alpha)+1).$$

Proof : First suppose $rank\ \ M < lq_0(1+\alpha)(lq_0(1+\alpha)+1)$. Then there exists a non-zero vector $b = (b_0, b_1, \ldots, b_{lq_0(1+\alpha)(lq_0(1+\alpha)+1)-1})$ such that

$$M.b^T = 0 \tag{6.4.2}$$

Set

$$\begin{aligned} f(z) &= \sum_{N=0}^{\infty} a_{N+(p+q_0c)l} z^N \\ &= \left\{b_0 + b_1 z + \ldots + b_{lq_0(1+\alpha)(lq_0(1+\alpha)+1)-1} z^{lq_0(1+\alpha)(lq_0(1+\alpha)+1)-1}\right\} \times \\ &\qquad \left\{1 - \left(\frac{z}{\rho}\right)^{lq_0(1+\alpha)(lq_0(1+\alpha)+1)}\right\}^{-1}. \end{aligned}$$

Clearly $f \in A_\rho$ and that

$$a_{N+(p+q_0c)l} = b_k \rho^{-lq_0(1+\alpha)(lq_0(1+\alpha)+1)\nu} \tag{6.4.3}$$

where $N = lq_0(1+\alpha)(lq_0(1+\alpha)+1)\nu + k$, $k = 0, 1, \ldots, lq_0(1+\alpha)(lq_0(1+\alpha)+1) - 1, \nu = 0, 1, \ldots$.

From (6.4.2) and (6.4.3), we have

$$\sum_{k=0}^{lq_0(1+\alpha)-1} a_{lq_0(1+\alpha)i+k+(p+q_0c)l} z_j^k = 0 \qquad \text{for each i and } j = 1, 2, \ldots, \mu. \tag{6.4.4.}$$

and

$$\sum_{k=0}^{lq_0(1+\alpha)} a_{(lq_0(1+\alpha)+1)i+k+(p+q_0c)l} z_j^k = 0 \qquad \text{for each i and } j = \mu+1, \ldots, s. \tag{6.4.5}$$

For any integer $i > 0$ let r and t be determined by

$$lq_0(1+\alpha)i + t = (lq_0(1+\alpha)+1)r, \qquad 0 \le t < lq_0(1+\alpha)+1$$

then for $j \ge \mu + 1$ from (6.4.5)

$$\sum_{k=0}^{i-1} a_{k+lq_0(1+\alpha)i+(p+q_0c)l} z_j^k = \sum_{k=0}^{t-1} a_{k+lq_0(1+\alpha)i+(p+q_0c)l} z_j^k + \sum_{k=t}^{i-1} a_{k+lq_0(1+\alpha)i+(p+q_0c)l} z_j^k$$

$$
\begin{aligned}
&= \sum_{k=0}^{t-1} a_{k+lq_0(1+\alpha)i+(p+q_0c)l} z_j^k + \\
&\quad \Big(a_{t+lq_0(1+\alpha)i+(p+q_0c)l} z_j^t + a_{t+1+lq_0(1+\alpha)i+(p+q_0c)l} z_j^{t+1} \\
&\quad + \ldots + a_{t+lq_0(1+\alpha)+lq_0(1+\alpha)i+(p+q_0c)l} z_j^{t+lq_0(1+\alpha)} \Big) + \\
&\quad + \Big(a_{t+lq_0(1+\alpha)+1+lq_0(1+\alpha)i+(p+q_0c)l} z_j^{t+lq_0(1+\alpha)+1} + \ldots \\
&\quad + a_{t+2lq_0(1+\alpha)+1+lq_0(1+\alpha)i+(p+q_0c)l} z_j^{t+2lq_0(1+\alpha)+1} \Big) + \\
&\quad \ldots + \Big(a_{i-1-lq_0(1+\alpha)+lq_0(1+\alpha)i+(p+q_0c)l} z_j^{i-1-lq_0(1+\alpha)} \\
&\quad \ldots + a_{i-1+lq_0(1+\alpha)i+(p+q_0c)l} z_j^{i-1} \Big) \\
&= \sum_{k=0}^{t-1} a_{k+lq_0(1+\alpha)i+(p+q_0c)l} z_j^k + \\
&\quad + \sum_{k=0}^{lq_0(1+\alpha)} a_{(lq_0(1+\alpha)+1)r+k+(p+q_0c)l} + z_j^{(lq_0(1+\alpha)+1)r+k-lq_0(1+\alpha)i} \\
&\quad + \sum_{k=0}^{lq_0(1+\alpha)} a_{(lq_0(1+\alpha)+1)(r+1)+k+(p+q_0c)l} z_j^{(lq_0(1+\alpha)+1)(r+1)+k-lq_0(1+\alpha)i} + \ldots \\
&\quad + \sum_{k=0}^{lq_0(1+\alpha)} a_{(lq_0(1+\alpha)+1)(i-1)+k+(p+q_0c)l} z_j^{(lq_0(1+\alpha)+1)(i-1)+k-lq_0(1+\alpha)i} \\
&= \sum_{k=0}^{t-1} a_{k+lq_0(1+\alpha)i+(p+q_0c)l} z_j^k + \\
&\quad + \sum_{k=0}^{lq_0(1+\alpha)} \sum_{\nu=r}^{i-1} a_{(lq_0(1+\alpha)+1)\nu+k+(p+q_0c)l} z_j^{(lq_0(1+\alpha)+1)\nu+k-lq_0(1+\alpha)i} \\
&= \sum_{k=0}^{t-1} a_{k+lq_0(1+\alpha)i+(p+q_0c)l} z_j^k + 0 \qquad \text{(from 6.4.5)} \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{lq_0(1+\alpha)i}} \right). \qquad \text{(for large } i\text{)}
\end{aligned}
$$

This for $\mu < j \leq s$ gives

$$
\begin{aligned}
\Delta^{q_0,p}_{i,n_i,l}(z_j; f) &= \sum_{t=l}^{\infty} \sum_{k=0}^{i-1} a_{k+tq_i} z_j^k \\
&= \sum_{t=l}^{\infty} \sum_{k=0}^{i-1} a_{k+t(q_0(1+\alpha)i+(p+q_0c))} z_j^k \\
&= \sum_{k=0}^{i-1} a_{k+lq_0(1+\alpha)i+(p+q_0c)l} z_j^k + \sum_{t=l+1}^{\infty} \sum_{k=0}^{i-1} a_{k+t(q_0(1+\alpha)i+(p+q_0c))} z_j^k \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{lq_0(1+\alpha)i}} \right) + \mathcal{O}\left(\frac{|z_j|^i}{(\rho-\epsilon)^{i(q_0(l+1)(1+\alpha)+1)}} \right). \qquad (6.4.6)
\end{aligned}
$$

From (2.3.8) and (2.3.9) by putting $l = \beta$ and $m = lq_0(1+\alpha)$ we find that for $|z| > \rho$, by

choosing ϵ suuficiently small, we can find $\eta > 0$ such that

$$\frac{1}{(\rho-\epsilon)^{lq_0(1+\alpha)\imath}} < \left(\frac{|z_\jmath|}{\rho^{lq_0(1+\alpha)+1}} - \eta\right)^\imath \tag{6.4.7}$$

and

$$\frac{|z_\jmath|}{(\rho-\epsilon)^{((l+1)q_0(1+\alpha)+1)\imath}} < \left(\frac{|z_\jmath|}{\rho^{lq_0(1+\alpha)+1}} - \eta\right)^\imath. \tag{6.4.8}$$

From (6.4.6), (6.4.7) and (6.4.8) we have

$$\Delta^{q_0,p}_{\imath,n_\imath,l}(z_\jmath; f) = \mathcal{O}\left(\frac{|z_\jmath|}{\rho^{lq_0(1+\alpha)+1}} - \eta\right)^\imath \qquad \text{for} \qquad |z_\jmath| > \rho. \tag{6.4.9}$$

Here and elsewhere η will denote sufficeintly small positive number which is not same at each occurence. Now, let for any integer $i > 0$, r and t be determined by

$$lq_0(1+\alpha)r + t = (lq_0(1+\alpha)+1)i, \qquad 0 \le t < lq_0(1+\alpha).$$

Then for $0 \le j \le \mu$, proceeding as before, from (6.4.4) we have

$$\begin{aligned}
\sum_{k=0}^{\imath-1} a_{k+lq_0(1+\alpha)\imath+(p+q_0c)l} z_j^k &= \sum_{k=lq_0(1+\alpha)\imath}^{lq_0(1+\alpha)i+\imath-1} a_{k+(p+q_0c)l} z_\jmath^{k-\imath lq_0(1+\alpha)} \\
&= \sum_{k=lq_0(1+\alpha)\imath}^{rlq_0(1+\alpha)-1} a_{k+(p+q_0c)l} z_\jmath^{k-\imath lq_0(1+\alpha)} + \\
&\quad + \sum_{k=rlq_0(1+\alpha)}^{(lq_0(1+\alpha)+1)\imath-1} a_{k+(p+q_0c)l} z_\jmath^{k-\imath lq_0(1+\alpha)} \\
&= \sum_{\nu=\imath}^{r-1} \sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq_0(1+\alpha)\nu+(p+q_0c)l} z_j^{k+lq_0(1+\alpha)(\nu-\imath)} + \\
&\quad + \sum_{k=rlq_0(1+\alpha)}^{(lq_0(1+\alpha)+1)\imath-1} a_{k+(p+q_0c)l} z_\jmath^{k-lq_0(1+\alpha)\imath} \\
&= 0 + \sum_{k=0}^{t-1} a_{k+(p+q_0c)l+rlq_0(1+\alpha)} z_\jmath^{k+lq_0(1+\alpha)(r-\imath)} \qquad \text{(from (6.4.4))} \\
&= \mathcal{O}\left(\frac{|z_\jmath|^{lq_0(1+\alpha)(r-\imath)}}{(\rho-\epsilon)^{rlq_0(1+\alpha)}}\right) \\
&= \mathcal{O}\left(\frac{|z_\jmath|^\imath}{(\rho-\epsilon)^{(lq_0(1+\alpha)+1)\imath}}\right)
\end{aligned}$$

whence for $0 \le j \le \mu$ we have

$$\begin{aligned}
\Delta^{q_0,p}_{\imath,n_\imath,1}(z_\jmath; f) &= \sum_{k=0}^{\imath-1} a_{k+lq_0(1+\alpha)\imath+(p+q_0c)l} z_j^k + \sum_{t=l+1}^{\infty} \sum_{k=0}^{\imath-1} a_{k+t(q_0(1+\alpha)\imath+(p+q_0c))} z_j^k \\
&= \mathcal{O}\left(\frac{|z_\jmath|^i}{(\rho-\epsilon)^{(lq_0(1+\alpha)+1)\imath}} + \frac{1}{(\rho-\epsilon)^{q_0(l+1)(1+\alpha)\imath}}\right).
\end{aligned} \tag{6.4.10}$$

From (2.3.21) and (2.3.22) for by putting $l=\beta$ and $m=lq_0(1+\alpha)$ we find that for $|z|<\rho$, by choosing ϵ suuficiently small, we can find $\eta>0$ such that

$$\frac{|z_j|^i}{(\rho-\epsilon)^{(1+lq_0(1+\alpha))i}} < \left(\frac{1}{\rho^{lq_0(1+\alpha)}}-\eta\right)^i \tag{6.4.11}$$

and

$$\frac{1}{(\rho-\epsilon)^{(l+1)q_0(1+\alpha)i}} < \left(\frac{1}{\rho^{lq_0(1+\alpha)}}-\eta\right)^i. \tag{6.4.12}$$

From (6.4.10), (6.4.11) and (6.4.12) we have

$$\Delta^{q_0,p}_{i,n_i,l}(z_j;f)=\mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)}}-\eta\right)^i \qquad \text{for} \qquad |z_j|<\rho. \tag{6.4.13}$$

Hence (6.4.9) and (6.4.13) gives

$$H_{l,q_0,\alpha}(z_j;f)<K_{l,q_0,\alpha}(|z_j|,\rho).$$

For the convers part suppose $H_{l,q_0,\alpha}(z_j;f)<K_{l,q_0,\alpha}(|z_j|,\rho)$ $(j=1,2,\ldots,s)$ for some $f(z)=\sum_{k=0}^{\infty}a_kz^k\in A_\rho$ and that $rank\ M=lq_0(1+\alpha)(lq_0(1+\alpha)+1)$. Set

$$\begin{aligned}
h(z) &= \Delta^{q_0,p}_{i,n_i,l}(z;f)-z^{lq_0(1+\alpha)}\Delta^{q_0,p}_{i+1,n_i+\alpha,l}(z;f)\\
&= \sum_{j=l}^{\infty}\sum_{k=0}^{i-1}a_{j((i+n_i)q_0+p)+k}z^k - z^{l(1+\alpha)q_0}\sum_{j=l}^{\infty}\sum_{k=0}^{i}a_{j((i+1+n_i+\alpha)q_0+p)+k}z^k\\
&= \sum_{k=0}^{i-1}a_{l((i+n_i)q_0+p)+k}z^k-\sum_{k=0}^{i}a_{l((i+1+n_i+\alpha)q_0+p)+k}z^{k+l(1+\alpha)q_0}+\\
&\quad+\sum_{j=l+1}^{\infty}\sum_{k=0}^{i-1}a_{j((i+n_i)q_0+p)+k}z^k-\sum_{j=l+1}^{\infty}\sum_{k=0}^{i}a_{j((i+n_i)q_0+(1+\alpha)q_0+p)+k}z^{k-l(1+\alpha)q_0}\\
&= \sum_{k=0}^{i-1}a_{l((i+n_i)q_0+p)+k}z^k-\sum_{k=l(1+\alpha)q_0}^{i+l(1+\alpha)q_0}a_{l((i+n_i)q_0+p)+k}z^k+\\
&\quad+\sum_{j=l+1}^{\infty}\sum_{k=0}^{i-1}a_{j((i+n_i)q_0+p)+k}z^k-\sum_{j=l+1}^{\infty}\sum_{k=0}^{i}a_{j((i+n_i)q_0+(1+\alpha)q_0+p)+k}z^{k-l(1-\alpha)q_0}\\
&= \left(\sum_{k=0}^{l(1+\alpha)q_0-1}+\sum_{k=l(1+\alpha)q_0}^{i-1}-\sum_{k=l(1+\alpha)q_0}^{i-1}-\sum_{k=i}^{i+l(1+\alpha)q_0}\right)a_{l(i+n_i)q_0+pl+k}z^k+\\
&\quad+\mathcal{O}((K_{l+1,q_0,\alpha}(|z|,\rho-\epsilon))^i)\\
&= \sum_{k=0}^{lq_0(1+\alpha)-1}a_{k+lq}z^k-\sum_{k=0}^{lq_0(1+\alpha)}a_{k+lq+i}z^{k+i}+\mathcal{O}((K_{l+1,q_0,\alpha}(|z|,\rho-\epsilon))^i). \qquad (6.4.14)
\end{aligned}$$

For $0\le j\le\mu$ from (6.4.11) and (6.4.12)

$$h(z_j) = \sum_{k=0}^{lq_0(1+\alpha)-1}a_{k+lq}z_j^k+$$

$$+\mathcal{O}\left(\frac{|z_j|^i}{(\rho-\epsilon)^{(lq_0(1+\alpha)+1)i}}+\frac{1}{(\rho-\epsilon)^{(l+1)q_0(1+\alpha)i}}\right)$$

$$= \sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq} z_j^k + \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)}}-\eta\right)^i. \tag{6.4.15}$$

Now from the hypothesis $H_{l,q_0,\alpha}(z_j;f) < K_{l,q_0,\alpha}(|z_j|,\rho)$ $(j=1,2,\ldots,\mu)$. That is

$$\overline{\lim_{i\to\infty}}|\Delta_{i,n_i,l}(z;f)|^{1/i} = \frac{1}{\rho^{lq_0(1+\alpha)}}-\eta$$

for some $\eta > 0$. Thus,

$$\Delta^{q_0,p}_{i,n_i,l}(z_j;f) \le \left(\frac{1}{\rho^{lq_0(1+\alpha)}}-\eta+\epsilon\right)^i$$

for $i \ge i_0(\epsilon)$ and $\eta > \epsilon > 0$. Thus,

$$\begin{aligned} h(z_j) &= \Delta^{q_0,p}_{i,n_i,l}(z_j;f) - z_j^{lq_0(1+\alpha)}\Delta^{q_0,p}_{i+1,n_i+\alpha,l}(z_j;f) \\ &= \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)}}-\eta\right)^i \end{aligned}$$

hence from (6.4.15) we obtain

$$\sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+l((i+n_i)q_0+p)} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)}}-\eta\right)^i. \tag{6.4.16}$$

Similarly for $j > \mu$ from (6.4.14) from (6.4.7) and (6.4.8) we have

$$\begin{aligned} h(z_j) &= -\sum_{k=0}^{lq_0(1+\alpha)} a_{k+lq+i} z_j^{k+i} + \\ &\quad +\mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{lq_0(1+\alpha)i}}+\frac{|z_j|^i}{(\rho-\epsilon)^{((l+1)(1+\alpha)q_0+1)i}}\right) \\ &= -\sum_{k=0}^{lq_0(1+\alpha)} a_{k+lq+i} z_j^{k+i} + \mathcal{O}\left(\frac{|z_j|}{\rho^{lq_0(1+\alpha)+1}}-\eta\right)^i. \end{aligned} \tag{6.4.17}$$

Now from the hypothesis $H_{l,q_0,\alpha}(z_j;f) < K_{l,q_0,\alpha}(|z_j|,\rho)$ $(j=\mu+1,\ldots,s)$. That is

$$\overline{\lim_{i\to\infty}}|\Delta^{q_0,p}_{i,n_i,l}(z_j;f)|^{1/i} = \frac{|z_j|}{\rho^{(lq_0(1+\alpha)+1)}}-\eta$$

for some $\eta > 0$. Thus,

$$\Delta^{q_0,p}_{i,n_i,l}(z_j;f) \le \left(\frac{|z_j|}{\rho^{lq_0(1+\alpha)+1}}-\eta+\epsilon\right)^i$$

for $i \ge i_0(\epsilon)$ and $\eta > \epsilon > 0$. Thus,

$$\begin{aligned} h(z_j) &= \Delta^{q_0,p}_{i,n_i,l}(z_j;f) - z_j^{lq_0(1+\alpha)}\Delta^{q_0,p}_{i+1,n_i+\alpha,l}(z_j;f) \\ &= \mathcal{O}\left(\frac{|z_j|}{\rho^{lq_0(1+\alpha)+1}}-\eta\right)^i \end{aligned}$$

hence from (6.4.17) we obtain

$$\sum_{k=0}^{lq_0(1+\alpha)} a_{k+lq+\imath} z_j^{k+\imath} = \mathcal{O}\left(\frac{|z_j|}{\rho^{lq_0(1+\alpha)+1}} - \eta\right)^{\imath}$$

or,

$$\sum_{k=0}^{lq_0(1+\alpha)} a_{k+l((\imath+n_\imath)q_0+p)+\imath} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)+1}} - \eta_1\right)^{\imath}. \tag{6.4.18}$$

Now, since (6.4.16) and (6.4.18) holds for all i, put $i = (lq_0(1+\alpha)+1)\nu+\lambda, \lambda = 0, \ldots, lq_0(1+\alpha)$ in (6.4.16) and $i = lq_0(1+\alpha)\nu + \lambda, \lambda = 0, \ldots, lq_0(1+\alpha) - 1$ in (6.4.18) we have

$$\sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq_0(1+\alpha)(lq_0(1+\alpha)+1)\nu+\lambda lq_0(1+\alpha)+(p+q_0c)l} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)}} - \eta\right)^{(lq_0(1+\alpha)+1)\nu+\lambda} \tag{6.4.19}$$

$(j = 1, \ldots, \mu; \lambda = 0, 1, \ldots lq_0(1+\alpha); \nu = 0, 1, \ldots)$ and

$$\sum_{k=0}^{lq_0(1+\alpha)} a_{k+(lq_0(1+\alpha)+1)lq_0(1+\alpha)\nu+\lambda(lq_0(1+\alpha)+1)+(p+q_0c)l} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)+1}} - \eta\right)^{lq_0(1+\alpha)\nu-\lambda} \tag{6.4.20}$$

$(j = \mu+1, \ldots, s; \lambda = 0, 1, \ldots lq_0(1+\alpha) - 1; \nu = 0, 1, \ldots)$.

Now since

$$\frac{1}{\rho^{lq_0(1+\alpha)}} - \eta < \frac{1}{\rho^{lq_0(1+\alpha)}}, \qquad \eta > 0$$

$$\left(\frac{1}{\rho^{lq_0(1+\alpha)}} - \eta\right)^{lq_0(1+\alpha)+1} < \frac{1}{\rho^{lq_0(1+\alpha)(lq_0(1-\alpha)-1)}}$$

choose η_1 such that

$$0 < \eta_1 < \frac{1}{\rho^{lq_0(1+\alpha)(lq_0(1+\alpha)+1)}} - \left(\frac{1}{\rho^{lq_0(1+\alpha)}} - \eta\right)^{lq_0(1+\alpha)+1}$$

or,

$$\left(\frac{1}{\rho^{lq_0(1+\alpha)}} - \eta\right)^{(lq_0(1+\alpha)+1)\nu} < \left(\frac{1}{\rho^{lq_0(1+\alpha)(lq_0(1-\alpha)+1)}} - \eta_1\right)^{\nu}$$

hence (6.4.19) can be written as

$$\sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq_0(1+\alpha)(lq_0(1+\alpha)+1)\nu+\lambda lq_0(1+\alpha)+(p+q_0c)l} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)(lq_0(1+\alpha)+1)}} - \eta\right)^{\nu} \tag{6.4.21}$$

$(j = 1, \ldots, \mu; \lambda = 0, 1, \ldots lq_0(1+\alpha); \nu = 0, 1, \ldots)$.

Similarly (6.4.20) can be written as

$$\sum_{k=0}^{lq_0(1+\alpha)} a_{k+(lq_0(1+\alpha)+1)lq_0(1+\alpha)\nu+\lambda(lq_0(1+\alpha)+1)+(p+q_0c)l} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)(lq_0(1+\alpha)+1)}} - \eta\right)^{\nu} \tag{6.4.22}$$

$(j = \mu+1, \ldots, s; \lambda = 0, 1, \ldots lq_0(1+\alpha)-1; \nu = 0, 1, \ldots)$.

Note that (6.4.21) and (6.4.22) can be written as

$$M.A^T = B \tag{6.4.23}$$

where

$$\begin{aligned} A \;=\; & \big(a_{lq_0(1+\alpha)(lq_0(1+\alpha)+1)\nu+(p+q_0c)l}, a_{lq_0(1+\alpha)(lq_0(1+\alpha)+1)\nu+(p+q_0c)l+1}, \ldots, \\ & \ldots, a_{lq_0(1+\alpha)(lq_0(1+\alpha)+1)\nu+(p+q_0c)l+lq_0(1+\alpha)(lq_0(1+\alpha)+1)-1}\big) \end{aligned}$$

and

$$B = \left(\mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)(lq_0(1+\alpha)+1)}} - \eta\right)^{\nu}\right),$$

B is a column vector of order $((slq_0(1+\alpha)+\mu)\times 1)$.

Since $rank\ M = lq_0(1+\alpha)(lq_0(1+\alpha)+1)$, solving (6.4.23) we get

$$a_{lq_0(1+\alpha)(lq_0(1+\alpha)+1)\nu+(p+q_0c)l+k} = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\alpha)(lq_0(1+\alpha)+1)}} - \eta\right)^{\nu}$$

for $k = 0, 1, \ldots, lq_0(1+\alpha)(lq_0(1+\alpha)+1)-1$. Hence

$$\overline{\lim_{\nu\to\infty}}|a_\nu|^{1/\nu} < \frac{1}{\rho}$$

which is a contradiction to $f \in A_\rho$.

Corollary 6.4.1 If either $\mu \geq lq_0(1+\alpha)$ or $s-\mu \geq lq_0(1+\alpha)+1$, then Z is not (l, q_0, α, ρ) distinguished.

If $\mu \geq lq_0(1+\alpha)$ then consider the minor of M consisting first $lq_0(1+\alpha)$ rows of each X. Determinant of this minor is $(van(1, z_1, z_2, \ldots, z_\mu))^{(lq_0(1+\alpha)-1)} \neq 0$. Obviously the number of rows in this minor is $lq_0(1+\alpha)(lq_0(1+\alpha)+1)$. Similar arguments holds for $s-\mu \geq lq_0(1+\alpha)+1$.

Corollary 6.4.2 If $\mu < s \leq lq_0(1+\alpha)$ or $\mu = s < lq_0(1+\alpha)$, then Z is (l, q_0, α, ρ) distinguished.

If $\mu < s \leq lq_0(1+\alpha)$ or $\mu = s < lq_0(1+\alpha)$, then number of rows $slq_0(1+\alpha)+\mu < lq_0(1+\alpha).lq_0(1+\alpha)+lq_0(1+\alpha) = lq_0(1+\alpha)(lq_0(1+\alpha)+1)$. Hence rank $M < lq_0(1+\alpha)(lq_0(1+\alpha)+1)$. From Corollary 6.4.1 we have

Theorem 6.4.2 *Let $f \in A_\rho, \rho > 1, \alpha \geq 0$ and $l \geq 1$. Then*

(i)
$$\overline{\lim_{i\to\infty}} |\Delta^{q_0,p}_{i,n_i,l}(z,f)|^{1/m_i} = \frac{|z|}{\rho^{lq_0(1+\alpha)+1}}$$

for all but at most $lq_0(1+\alpha)$ points in $|z| > \rho$.

(ii)
$$\overline{\lim_{i\to\infty}} |\Delta^{q_0,p}_{i,n_i,l}(z,f)|^{1/m_i} = \frac{1}{\rho^{lq_0(1+\alpha)}}$$

for all but at most $lq_0(1+\alpha) - 1$ points in $|z| < \rho$.

Corollary 6.4.2 implies that Theorem 6.4.2 cannot be improved. That is

Theorem 6.4.3 *Let $\rho > 1, \alpha \geq 0$ and $l \geq 1$.*

(i) If $z_1, \ldots, z_{lq_0(1+\alpha)}$ are arbitrary $lq_0(1+\alpha)$ points with modulus greater than ρ then there is a rational function $f \in A_\rho$ with

$$\overline{\lim_{i\to\infty}} |\Delta^{q_0,p}_{i,n_i,l}(z_j,f)|^{1/m_i} < \frac{|z_j|}{\rho^{lq_0(1+\alpha)+1}}, \qquad j = 1, \ldots, lq_0(1+\alpha).$$

(ii) If $z_1, \ldots, z_{lq_0(1+\alpha)-1}$ are arbitrary $lq_0(1+\alpha) - 1$ points in the ring $0 < |z| < \rho$ then there is a rational function $f \in A_\rho$ with

$$\overline{\lim_{i\to\infty}} |\Delta^{q_0,p}_{i,n_i,l}(z_j,f)|^{1/m_i} < \frac{1}{\rho^{lq_0(1+\alpha)}}, \qquad j = 1, \ldots, lq_0(1+\alpha) - 1.$$

6.5 In this section we study properties of a set containing points in $|z| < \rho$ and $|z| > \rho$ simultaneously, regarding the number of points in $|z| < \rho$ and $|z| > \rho$, by defining the concept of distinguished points for the polynomials in z^{-1}.

Let

$$m_i = i, \forall i \qquad \text{and} \qquad n_i = \alpha i, \forall i, \qquad \alpha \text{ is an integer} > 0. \tag{6.5.1}$$

If we set

$$G_{l,q_0,\alpha}(z^{-1}; f) = \overline{\lim_{i\to\infty}} |\Theta^{q_0,p}_{n_i,i,l}(z^{-1}; f)|^{1/n_i}$$

then from the result of Theorem 6.3.2 and the definition of $B_{l,q_0,\alpha}(|z^{-1}|, \rho)$ it follows that $G_{l,q_0,\alpha}(z^{-1}; f) \leq B_{l,q_0,\alpha}(|z^{-1}|, \rho)$. We shall say that a set $Z \subset C$ of points to be $(l, q_0, \alpha, \rho^{-1})$ distinguished if there is an $f \in A_\rho$ such that

$$G_{l,q_0,\alpha}(z_j^{-1}; f) < B_{l,q_0,\alpha}(|z_j^{-1}|, \rho),$$

for each $z_j \in Z$.

Suppose $Z = \{z_j\}_1^s$ is given in which $|z_j| < \rho$ $(j = 1, \ldots, \mu)$ and $|z_j| > \rho$ $(j = \mu + 1, \ldots, s)$.

We want to find a criterion to determine whether Z is $(l, q_0, \alpha, \rho^{-1})$ distinguished or not. Set the matrices X, Y, M (X, Y) as

$$X = \begin{pmatrix} 1 & z_1 & \dots & z_1^{lq_0(1+\alpha)-1-\alpha} \\ \dots & \dots & \dots & \dots \\ 1 & z_\mu & \dots & z_\mu^{lq_0(1+\alpha)-1-\alpha} \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & z_{\mu+1} & \dots & z_{\mu+1}^{lq_0(1-\alpha)-1} \\ \dots & \dots & \dots & \dots \\ 1 & z_s & \dots & z_s^{lq_0(1-\alpha)-1} \end{pmatrix}$$

The matrices X and Y are of order $(\mu \times (lq_0(1+\alpha)-\alpha))$ and $(s-\mu) \times (lq_0(1+\alpha))$ respectively. Define

$$M = M(X,Y) = \begin{pmatrix} X & & & \\ & X & & 0 \\ & & \ddots & \\ & 0 & & X \\ Y & & & \\ & Y & & 0 \\ & & \ddots & \\ & 0 & & Y \end{pmatrix},$$

where X occurs $lq_0(1+\alpha)$ times and Y occurs $lq_0(1+\alpha)-\alpha$ times beginning under the last X. The matrix M is of order $(s(lq_0(1+\alpha)-\alpha)+\mu\alpha) \times (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)$. We now formulate

Theorem 6.5.1 *Suppose $Z = \{z_j\}_1^s$ is a set of points in C such that $|z_j| < \rho$ $(j = 1, \dots, \mu)$ and $|z_j| > \rho$ $(j = \mu+1, \dots, s)$. Then the set Z is $(l, q_0, \alpha, \rho^{-1})$ distinguished iff*

$$rank \;\; M < (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha).$$

Proof : First suppose $rank \;\; M < (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)$. Then there exists a non-zero vector $b = (b_0, b_1, \dots, b_{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)-1})$ such that

$$M.b^T = 0 \tag{6.5.2}$$

Set

$$\begin{aligned} f(z) &= \sum_{N=0}^{\infty} a_{N+pl} z^N \\ &= \left\{ b_0 + b_1 z + \dots + b_{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)-1} z^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)-1} \right\} \times \\ &\qquad \left\{ 1 - \left(\frac{z}{\rho}\right)^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)} \right\}^{-1}. \end{aligned}$$

Clearly $f \in A_\rho$ and that

$$a_{N+pl} = b_k \rho^{-(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)\nu} \tag{6.5.3}$$

where $N = (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)\nu + k$, $k = 0, 1, \ldots, (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)-1, \nu = 0, 1, \ldots$.

From (6.5.2) and (6.5.3), we have

$$\sum_{k=0}^{lq_0(1+\alpha)-\alpha-1} a_{(lq_0(1+\alpha)-\alpha)i+k+pl} z_j^k = 0 \qquad \text{for each i and } j = 1, 2, \ldots, \mu. \tag{6.5.4}$$

and

$$\sum_{k=0}^{lq_0(1+\alpha)-1} a_{lq_0(1+\alpha)i+k+pl} z_j^k = 0 \qquad \text{for each i and } j = \mu+1, \ldots, s \tag{6.5.5}$$

For any integer $i > 0$ let r and t be determined by

$$(lq_0(1+\alpha)-\alpha)i + t = lq_0(1+\alpha)r, \qquad 0 \le t < lq_0(1+\alpha)$$

then for $j \ge \mu + 1$ from (6.5.5)

$$\begin{aligned}
\sum_{k=0}^{n_i-1} a_{k+lq_i-n_i} z^{k-n_i} &= \sum_{k=0}^{n_i-1} a_{k+(lq_0(1+\alpha)-\alpha)i+pl} z_j^{k-n_i} \\
&= \sum_{k=0}^{t-1} a_{k+(lq_0(1+\alpha)-\alpha)i+pl} z_j^{k-n_i} + \sum_{k=t}^{n_i-1} a_{k+(lq_0(1+\alpha)-\alpha)i+pl} z_j^{k-n_i} \\
&= \sum_{k=0}^{t-1} a_{k+lq_0(1+\alpha)i+pl} z_j^{k-n_i} + \\
&\quad + \sum_{k=0}^{lq_0(1+\alpha)-1} a_{lq_0(1+\alpha)r+k+pl} z_j^{lq_0(1+\alpha)r+k-(lq_0(1+\alpha)-\alpha)i-n_i} \\
&\quad + \sum_{k=0}^{lq_0(1+\alpha)-1} a_{lq_0(1+\alpha)(r+1)+k+pl} z_j^{lq_0(1+\alpha)(r+1)+k-(lq_0(1+\alpha)-\alpha)i-n_i} + \ldots \\
&\quad + \sum_{k=0}^{lq_0(1+\alpha)-1} a_{lq_0(1+\alpha)(i-1)+k+pl} z_j^{lq_0(1+\alpha)(i-1)+k-(lq_0(1+\alpha)-\alpha)i-n_i} \\
&= \sum_{k=0}^{t-1} a_{k+(lq_0(1+\alpha)-\alpha)i+pl} z_j^{k-n_i} + \\
&\quad \sum_{k=0}^{lq_0(1+\alpha)-1} \sum_{\nu=r}^{i-1} a_{lq_0(1+\alpha)\nu+k+pl} z_j^{lq_0(1+\alpha)\nu+k-(lq_0(1+\alpha)-\alpha)i-n_i} \\
&= \sum_{k=0}^{t-1} a_{k+(lq_0(1+\alpha)-\alpha)i+pl} z_j^{k-n_i} + 0 \qquad \text{(from (6.5.5))} \\
&= \mathcal{O}\left(\frac{|z_j|^{-n_i}}{(\rho-\epsilon)^{(lq_0(1+\alpha)-\alpha)i}}\right) \qquad \text{(for large } i) \\
&= \mathcal{O}\left(\frac{|z_j|^{-n_i}}{(\rho-\epsilon)^{(lq_0(1+\frac{1}{\alpha})-1)n_i}}\right).
\end{aligned}$$

This for $\mu < j \le s$ gives

$$\begin{aligned}
\Theta^{q_0,p}_{n_t,t,l}(z_j^{-1};f) &= \sum_{b=l}^{\infty}\sum_{k=0}^{n_t-1} a_{k+bq_t-n_t} z_j^{k-n_t} \\
&= \sum_{k=0}^{n_t-1} a_{k+lq_t-n_t} z_j^{k-n_t} + \sum_{b=l+1}^{\infty}\sum_{k=0}^{n_t-1} a_{k+bq_t-n_t} z_j^{k-n_t} \\
&= \mathcal{O}\left(\frac{|z_j|^{-n_t}}{(\rho-\epsilon)^{lq_0(1+\frac{1}{\alpha})-1)n_t}}\right) + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{q_0(l+1)(1+\frac{1}{\alpha}n_t}}\right). \qquad (6.5.6)
\end{aligned}$$

Now choose $\epsilon_1 > 0$ so small that

$$\frac{|z_j|^{-1}}{(\rho-\epsilon_1)^{lq_0(1+\frac{1}{\alpha})-1}} < \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} \qquad |z_j| > \rho.$$

Choose $\eta_1 > 0$ such that

$$0 < \eta_1 < \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \frac{|z_j|^{-1}}{(\rho-\epsilon_1)^{lq_0(1+\frac{1}{\alpha})-1}}. \qquad (6.5.7)$$

Similarly choose $\epsilon_2 > 0$ so small that

$$\frac{1}{(\rho-\epsilon_2)^{(l+1)q_0(1+\frac{1}{\alpha})}} < \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} \qquad |z_j| > \rho$$

and choose $\eta_2 > 0$ such that

$$0 < \eta_2 < \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \frac{1}{(\rho-\epsilon_2)^{(l+1)q_0(1+\frac{1}{\alpha})}}. \qquad (6.5.8)$$

Let

$$\epsilon = min(\epsilon_1,\epsilon_2) \qquad \text{and} \qquad \eta = min(\eta_1,\eta_2).$$

From (6.5.7) we have

$$\frac{|z_j|^{-n}}{(\rho-\epsilon)^{(lq_0(1+\frac{1}{\alpha})-1)n}} < \left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \eta\right)^n. \qquad (6.5.9)$$

Similarly from (6.5.8) we have

$$\frac{1}{(\rho-\epsilon)^{(l+1)q_0(1+\frac{1}{\alpha})n_t}} < \left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \eta\right)^{n_t}. \qquad (6.5.10)$$

From (6.5.6), (6.5.9) and (6.5.10) we have

$$\Theta^{q_0,p}_{n_t,t,,l}(z_j^{-1};f) = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \eta\right)^{n_t} \qquad \text{for} \qquad |z_j| > \rho. \qquad (6.5.11)$$

Now, let for any integer $i > 0$, r and t be determined by

$$(lq_0(1+\alpha)-\alpha)r + t = lq_0(1+\alpha)i, \qquad 0 \le t < lq_0(1+\alpha)-\alpha.$$

Then for $0 \leq j \leq \mu$, proceeding as before, from (6.5.4) we have

$$\begin{aligned}
\sum_{k=0}^{n_i-1} a_{k+lq_i-n_i} z^{k-n_i} &= \sum_{k=0}^{n_i-1} a_{k+(lq_0(1+\alpha)-\alpha)i+pl} z_j^{k-n_i} \\
&= \sum_{k=(lq_0(1+\alpha)-\alpha)i}^{(lq_0(1+\alpha)-\alpha)i+n_i-1} a_{k+pl} z_j^{k-(lq_0(1+\alpha)-\alpha)i-n_i} \\
&= \sum_{k=(lq_0(1+\alpha)-\alpha)i}^{(lq_0(1+\alpha)-\alpha)r-1} a_{k+pl} z_j^{k-(lq_0(1+\alpha)-\alpha)i-n_i} + \\
&\quad + \sum_{k=r(lq_0(1+\alpha)-\alpha)}^{(lq_0(1+\alpha)-\alpha)i-n_i-1} a_{k+pl} z_j^{k-(lq_0(1+\alpha)-\alpha)i-n_i} \\
&= \sum_{\nu=i}^{r-1} \sum_{k=0}^{lq_0(1+\alpha)-\alpha-1} a_{k+(lq_0(1+\alpha)-\alpha)\nu+pl} z_j^{k+(lq_0(1-\alpha)-\alpha)(\nu-i)-n_i} \\
&\quad + \sum_{k=r(lq_0(1+\alpha)-\alpha)}^{(lq_0(1+\alpha)-\alpha)i+n_i-1} a_{k+pl} z_j^{k-lq_0(1+\alpha)i} \\
&= 0 + \sum_{k=0}^{t-1} a_{k+pl+r(lq_0(1+\alpha)-\alpha)} z_j^{k+(lq_0(1+\alpha)-\alpha)(r-i)-n_i} \qquad \text{(from (6.5.4)} \\
&= \mathcal{O}\left(\frac{|z_j|^{(lq_0(1+\alpha)-\alpha)(r-i)-n_i}}{(\rho-\epsilon)^{r(lq_0(1+\alpha)-\alpha)}}\right) \\
&= \mathcal{O}\left(\frac{|z_j|^{\alpha n-n_i}}{(\rho-\epsilon)^{lq_0(1+\alpha)i}}\right) \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{lq_0(1+\frac{1}{\alpha})n_i}}\right).
\end{aligned}$$

Hence for $0 \leq j \leq \mu$ we have

$$\begin{aligned}
\Theta_{n_i,i,1}^{q_0,p}(z_j^{-1}; f) &= \sum_{k=0}^{n_i-1} a_{k+lq_i-n_i} z_j^{k-n_i} + \sum_{b=l+1}^{\infty} \sum_{k=0}^{n_i-1} a_{k+bq_i-n_i} z_j^{k-n_i} \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{lq_0(1+\frac{1}{\alpha})n_i}} + \frac{|z_j|^{-n_i}}{(\rho-\epsilon)^{(q_0(l-1)(1+\frac{1}{\alpha})-1)n_i}}\right) \qquad (6.5.12)
\end{aligned}$$

Proceeding as before we can show that for $|z| < \rho$ by choosing ϵ sufficiently small we can find $\eta > 0$ such that

$$\frac{1}{(\rho-\epsilon)^{lq_0(1+\frac{1}{\alpha})n_i}} < \left(\frac{|z_j|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{n_i} \qquad (6.5.13)$$

and

$$\frac{|z_j|^{-1}}{(\rho-\epsilon)^{((l+1)q_0(1+\frac{1}{\alpha})-1)n_i}} < \left(\frac{|z_j|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{n_i}. \qquad (6.5.14)$$

From (6.5.12), (6.5.13) and (6.5.14) we have

$$\Theta_{n_i,i,l}^{q_0,p}(z_j^{-1}; f) = \mathcal{O}\left(\frac{|z_j|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{n_i} \qquad \text{for} \qquad |z_j| < \rho. \qquad (6.5.15)$$

Hence (6.5.11) and (6.5.15) gives

$$G_{l,q_0,\alpha}(z_j^{-1};f) < B_{l,q_0,\alpha}(|z_j|^{-1},\rho).$$

For the convers part suppose $G_{l,q_0,\alpha}(z_j^{-1};f) < B_{l,q_0,\alpha}(|z_j|^{-1},\rho)$ $(j=1,2,\ldots,s)$ for some $f \in A_\rho$ and that *rank* $M = (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)$. Set

$$\begin{aligned}
h(z^{-1}) &= \Theta^{q_0,p}_{n_i,i,l}(z^{-1};f) - z^{lq_0(1+\alpha)-\alpha}\Theta^{q_0,p}_{n_i+\alpha,i+1,l}(z^{-1};f) \\
&= \sum_{j=l}^{\infty}\sum_{k=0}^{n_i-1} a_{j((i+n_i)q_0+p)+k-n_i} z^{k-n_i} - \\
&\quad - z^{l(1+\alpha)q_0}\sum_{j=l}^{\infty}\sum_{k=0}^{n_i+\alpha-1} a_{j((i+1+n_i+\alpha)q_0+p)+k-n_i-\alpha} z^{k-n_i-\alpha} \\
&= \sum_{k=0}^{n_i-1} a_{l((i+n_i)q_0+p)+k-n_i} z^{k-n_i} - \\
&\quad - \sum_{k=0}^{n_i+\alpha-1} a_{l((i+1+n_i+\alpha)q_0+p)+k-n_i-\alpha} z^{k+l(1+\alpha)q_0-n_i-\alpha} + \\
&\quad + \sum_{j=l+1}^{\infty}\sum_{k=0}^{n_i-1} a_{j((i+n_i)q_0+p)+k-n_i} z^{k-n_i} - \\
&\quad - \sum_{j=l+1}^{\infty}\sum_{k=0}^{n_i+\alpha-1} a_{j((i+n_i)q_0+(1+\alpha)q_0+p)+k-n_i-\alpha} z^{k+l(1+\alpha)q_0-n_i-\alpha} \\
&= \sum_{k=0}^{n_i-1} a_{l((i+n_i)q_0+p)+k-n_i} z^{k-n_i} - \sum_{k=l(1+\alpha)q_0-\alpha}^{n_i+l(1+\alpha)q_0-1} a_{l((i+n_i)q_0+p)+k-n_i} z^{k-n_i} + \\
&\quad + \sum_{j=l+1}^{\infty}\sum_{k=0}^{n_i-1} a_{j((i+n_i)q_0+p)+k-n_i} z^{k-n_i} \\
&\quad - \sum_{j=l+1}^{\infty}\sum_{k=0}^{n_i+\alpha-1} a_{j((i+n_i)q_0+(1+\alpha)q_0+p)+k-n_i-\alpha} z^{k+l(1+\alpha)q_0-n_i-\alpha} \\
&= \left(\sum_{k=0}^{l(1+\alpha)q_0-\alpha-1} + \sum_{k=l(1+\alpha)q_0-\alpha}^{n_i-1} - \sum_{k=l(1-\alpha)q_0-\alpha}^{n_i-1} - \right. \\
&\quad \left. \sum_{k=n_i}^{n_i+l(1+\alpha)q_0-1}\right) a_{l(i+n_i)q_0+pl+k-n_i} z^{k-n_i} + \mathcal{O}((B_{l-1,q_0,\alpha}(|z|^{-1},\rho-\epsilon))^{n_i}) \\
&= \sum_{k=0}^{lq_0(1+\alpha)-\alpha-1} a_{k+(lq_0(1+\alpha)-\alpha)i+lpl} z^{k-n_i} - \sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq_0(1+\alpha)i-lp} z^{k} + \\
&\quad + \mathcal{O}((B_{l+1,q_0,\alpha}(\rho-\epsilon,z))^{n_i}). \qquad (6.5.16)
\end{aligned}$$

For $0 \le j \le \mu$ from (6.5.13) and (6.5.14)

$$h(z_j^{-1}) = \sum_{k=0}^{lq_0(1+\alpha)-\alpha-1} a_{k+(lq_0(1+\alpha)-\alpha)i+lp} z_j^{k-n_i} +$$

$$+\mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{(lq_0(1+\frac{1}{\alpha})n_\imath}}+\frac{|z_\jmath|^{-n_\imath}}{(\rho-\epsilon)^{(l+1)q_0(1+\frac{1}{\alpha})n_\imath}}\right)$$

$$= \sum_{k=0}^{lq_0(1+\alpha)-\alpha-1} a_{k+(lq_0(1+\alpha)-\alpha)\imath+lp} z_\jmath^{k-n_\imath} + \mathcal{O}\left(\frac{|z_\jmath|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}}-\eta\right)^{n_\imath}. \qquad (6.5.17)$$

Now from the hypothesis $G_{l,q_0,\alpha}(z_\jmath^{-1};f) < B_{l,q_0,\alpha}(|z_\jmath|^{-1},\rho)$ $(\jmath = 1,2,\ldots,\mu)$. That is

$$\overline{\lim_{n_\imath\to\infty}}|\Theta^{q_0,p}_{n_\imath,\imath,l}(z_\jmath^{-1};f)|^{1/n_\imath} = \frac{|z_\jmath|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta.$$

for some $\eta > 0$. Thus,

$$\Theta^{q_0,p}_{n_\imath,\imath,l}(z_\jmath^{-1};f) \le \left(\frac{|z_\jmath|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{n_\imath}$$

for $n_\imath \ge n_0(\epsilon), 0 < \epsilon < \eta > 0$. Thus,

$$\begin{aligned} h(z_\jmath^{-1}) &= \Theta^{q_0,p}_{n_\imath,\imath,l}(z_\jmath^{-1};f) - z_\jmath^{lq_0(1+\alpha)}\Theta^{q_0,p}_{n_\imath+\alpha,\imath+1,l}(z_\jmath^{-1};f) \\ &= \mathcal{O}\left(\frac{|z_\jmath|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{n_\imath} \end{aligned}$$

hence from (6.5.17) we obtain

$$\sum_{k=0}^{lq_0(1+\alpha)-\alpha-1} a_{k+(l(1+\alpha)q_0-\alpha)i+lp} z_\jmath^{k} = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta_1\right)^{n_\imath}. \qquad (6.5.18)$$

Similarly for $\jmath > \mu$ from (6.5.16), (6.5.9) and (6.5.10) we have

$$\begin{aligned} h(z_\jmath^{-1}) &= -\sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq_0(1+\alpha)\imath+lp} z_\jmath^{k} + \\ &\quad +\mathcal{O}\left(\frac{|z_\jmath|^{-n_\imath}}{(\rho-\epsilon)^{(lq_0(1+\frac{1}{\alpha})-1)n_\imath}}+\frac{1}{(\rho-\epsilon)^{(l+1)q_0(1+\frac{1}{\alpha})n_\imath}}\right) \\ &= -\sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq_0(1+\alpha)\imath+lp} z_\jmath^{k} + \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \eta\right)^{n_\imath}. \end{aligned} \qquad (6.5.19)$$

Now from the hypothesis $G_{l,q_0,\alpha}(z_\jmath^{-1};f) < B_{l,q_0,\alpha}(|z_\jmath|^{-1},\rho)$ $(\jmath = \mu+1,\ldots,s)$. That is

$$\overline{\lim_{n_\imath\to\infty}}|\Theta^{q_0,p}_{n_\imath,\imath,l}(z_\jmath^{-1};f)|^{1/n_\imath} = \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \eta.$$

for some $\eta > 0$. Thus,

$$\Theta^{q_0,p}_{n_\imath,\imath,l}(z_\jmath^{-1};f) \le \left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \eta\right)^{n_\imath}$$

for $n_\imath \ge n_0(\epsilon), 0 < \epsilon < \eta$. Thus

$$\begin{aligned} h(z_\jmath^{-1}) &= \Theta^{q_0,p}_{n_\imath,\imath,l}(z_\jmath^{-1};f) - z_\jmath^{lq_0(1+\alpha)}\Theta^{q_0,p}_{n_\imath+\alpha,\imath+1,l}(z_\jmath^{-1};f) \\ &= \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \eta\right)^{n_\imath} \end{aligned}$$

hence from (6.5.19) we obtain

$$\sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+l(1+\alpha)q_0 i+lp} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}} - \eta\right)^{n_i}. \tag{6.5.20}$$

Now, since (6.5.18) and (6.5.20) holds for all i, put $i = lq_0(1+\alpha)\nu+\lambda, \lambda = 0, \ldots, lq_0(1+\alpha)-1$ in (6.5.18) and $i = (lq_0(1+\alpha)-\alpha)\nu + \lambda, \lambda = 0, \ldots, lq_0(1+\alpha)-\alpha-1$ in (6.5.20) we have

$$\sum_{k=0}^{lq_0(1+\alpha)-\alpha-1} a_{k+(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)\nu+\lambda(lq_0(1+\alpha)-\alpha)+pl} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{\alpha(lq_0(1+\alpha)\nu+\lambda)} \tag{6.5.21}$$

$(j = 1, \ldots, \mu; \lambda = 0, 1, \ldots lq_0(1+\alpha)-1; \nu = 0, 1, \ldots)$,

and

$$\sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq_0(1+\alpha)(lq_0(1+\alpha)-\alpha)\nu+\lambda lq_0(1+\alpha)+pl} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})+1}} - \eta\right)^{\alpha((lq_0(1+\alpha)-\alpha)\nu+\lambda)} \tag{6.5.22}$$

$(j = \mu+1, \ldots, s; \lambda = 0, 1, \ldots lq_0(1+\alpha)-\alpha-1; \nu = 0, 1, \ldots)$.

Now since

$$\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta < \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})-1}}, \qquad \eta > 0$$

$$\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{\alpha lq_0(1+\alpha)} < \frac{1}{\rho^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)}}$$

choose η_1 such that

$$0 < \eta_1 < \frac{1}{\rho^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)}} - \left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{\alpha lq_0(1-\alpha)}$$

or,

$$\left(\frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})-1}} - \eta\right)^{\alpha lq_0(1+\alpha)\nu} < \left(\frac{1}{\rho^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)}} - \eta_1\right)^{\nu}$$

hence (6.5.21) can be written as

$$\sum_{k=0}^{lq_0(1+\alpha)-\alpha-1} a_{k+(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)\nu+\lambda(lq_0(1+\alpha)-\alpha)+(p+q_0c)l} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)}} - \eta\right)^{\nu} \tag{6.5.23}$$

$(j = 1, \ldots, \mu; \lambda = 0, 1, \ldots lq_0(1+\alpha)-1; \nu = 0, 1, \ldots)$.

Similarly (6.5.22) can be written as

$$\sum_{k=0}^{lq_0(1+\alpha)-1} a_{k+lq_0(1+\alpha)(lq_0(1+\alpha)-\alpha)\nu+\lambda lq_0(1+\alpha)+(p+q_0c)l} z_j^k = \mathcal{O}\left(\frac{1}{\rho^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)}} - \eta\right)^{\nu} \tag{6.5.24}$$

$(j = \mu+1, \ldots, s; \lambda = 0, 1, \ldots lq_0(1+\alpha)-\alpha-1; \nu = 0, 1, \ldots)$.

Note that (6.5.23) and (6.5.24) can be written as

$$M.A^T = B \tag{6.5.25}$$

where

$$A = (a_{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)\nu+pl}, a_{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)\nu+pl+1}, \ldots,$$
$$\ldots, a_{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)\nu+pl+(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)-1})$$

and

$$B = \left(\mathcal{O}\left(\frac{1}{\rho^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)}} - \eta\right)^{\nu}\right),$$

B is a column vector of order $((s(lq_0(1+\alpha)-\alpha)+\mu\alpha)\times 1)$.

Since $rank\ M = (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)$, solving (6.5.25) we get

$$a_{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)\nu+pl+k} = \mathcal{O}\left(\frac{1}{\rho^{(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)}} - \eta\right)^{\nu}$$

for $k = 0, 1, \ldots, (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)-1$. Hence

$$\overline{\lim_{\nu\to\infty}}|a_\nu|^{1/\nu} < \frac{1}{\rho}$$

which is a contradiction to $f \in A_\rho$.

Corollary 6.5.1 If either $\mu \geq lq_0(1+\alpha)-\alpha$ or $s-\mu \geq lq_0(1+\alpha)$, then Z is not $(l, q_0, \alpha, \rho^{-1})$ distinguished.

If $\mu \geq lq_0(1+\alpha)-\alpha$ then consider the minor of M consisting first $lq_0(1+\alpha)-\alpha$ rows of each X. Determinant of this minor is $(van(1, z_1, z_2, \ldots, z_\mu))^{lq_0(1+\alpha)} \neq 0$. Obviously the number of rows in this minor is $(lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)$. Similar arguments holds for $s-\mu \geq lq_0(1+\alpha)$.

Corollary 6.5.2 If $\mu < s \leq lq_0(1+\alpha)-\alpha$ or $\mu = s < lq_0(1+\alpha)-\alpha$, then Z is $(l, q_0, \alpha, \rho^{-1})$ distinguished.

If $\mu < s \leq lq_0(1+\alpha)-\alpha$ or $\mu = s < lq_0(1+\alpha)-\alpha$, then number of rows $(s(lq_0(1+\alpha)-\alpha)+\mu\alpha < (lq_0(1+\alpha)-\alpha)(lq_0(1+\alpha)-\alpha)+(lq_0(1+\alpha)-\alpha)\alpha = (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)$.
Hence rank $M < (lq_0(1+\alpha)-\alpha)lq_0(1+\alpha)$.

From Corollary 6.5.1 we have

Theorem 6.5.2 *Let $f \in A_\rho, \rho > 1, \alpha > 0$ and $l \geq 2$. Then*

$$(i) \qquad \overline{\lim_{i\to\infty}}|\Theta^{q_0,p}_{n_i,i,l}(z^{-1}, f)|^{1/n_i} = \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}}$$

for all but at most $lq_0(1+\alpha)-1$ *points in* $|z|>\rho$.

$$(ii) \qquad \overline{\lim_{\imath\to\infty}}|\Theta^{q_0,p}_{n_\imath,\imath,l}(z^{-1},f)|^{1/n_\imath} = \frac{|z_\jmath|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}}$$

for all but at most $lq_0(1+\alpha)-\alpha-1$ *points in* $|z|<\rho$.

Further Corollary 6.5.2 implies that Theorem 6.5.2 cannot be improvrd. That is

Theorem 6.5.3 *Let* $\rho>1, \alpha>0$ *and* $l\geq 2$.

(i) If $z_1,\ldots,z_{lq_0(1+\alpha)-1}$ *are arbitrary* $lq_0(1+\alpha)-1$ *points with modulus greater than* ρ *then there is a rational function* $f\in A_\rho$ *with*

$$\overline{\lim_{\imath\to\infty}}|\Theta^{q_0,p}_{n_\imath,\imath,l}(z_\jmath^{-1},f)|^{1/n_\imath} < \frac{1}{\rho^{lq_0(1+\frac{1}{\alpha})}}, \qquad j=1,\ldots,lq_0(1+\alpha)-1.$$

(ii) If $z_1,\ldots,z_{lq_0(1+\alpha)-\alpha-1}$ *are arbitrary* $lq_0(1+\alpha)-\alpha-1$ *points in the ring* $0<|z|<\rho$ *then there is a rational function* $f\in A_\rho$ *with*

$$\overline{\lim_{\imath\to\infty}}|\Theta^{q_0,p}_{n_\imath,\imath,l}(z_j^{-1},f)|^{1/n_\imath} < \frac{|z_j|^{-1}}{\rho^{lq_0(1+\frac{1}{\alpha})-1}}, \qquad j=1,\ldots,lq_0(1+\alpha)-\alpha-1.$$

Chapter 7

WALSH OVERCONVERGENCE OF FUNCTIONS ANALYTIC IN AN ELLIPSE

7.1 Mostly authors have considered the functions analytic inside a circle. Rivlin was the first to introduce the functions analytic inside an ellipse. Suppose $1 < \rho < \infty$. Let C_ρ be the ellipse, in the z-plane, which is the image of the circle $|w| = \rho$, in the w-plane, under the mapping

$$z = \frac{w + 1/w}{2}.$$

This mapping maps the exterior as well as the interior of $|w| = 1$ in a 1-1 conformal fashion on the (extended) z-plane with the interval $[-1, 1]$ deleted. Each pair of circles $|w| = \rho, 1/\rho$ is mapped onto the same ellipse in the z-plane, C_ρ, with foci at $(\pm 1, 0)$ and the sum of major and minor axis equal to 2ρ.

Let $A(C_\rho)$ denote the class of functions, f, analytic inside C_ρ and having a singularity on C_ρ. Let

$$f(z) = {\sum_{k=0}^{\infty}}' A_k T_k(z) \tag{7.1.1}$$

where $T_k(z) = (w^k + w^{-k})/2$ is the Chebyshev polynomial of degree k and the stroke on the summation sign means that the first term of the sum is to be halved and

$$A_k = \frac{2}{\pi} \int_\Gamma f\left(\frac{(w + w^{-1})}{2}\right)(w^k + w^{-k})\frac{dw}{w}. \tag{7.1.2}$$

where Γ is $|w| = R$.

Rivlin [38] showed that for $f \in A(C_\rho)$ and having expansion (7.1.1),

$$A_k = \mathcal{O}(\rho - \epsilon)^{-k}$$

for every ϵ satisfying $0 < \epsilon < \rho - 1$ and $k \geq k_0(\epsilon)$. Let $q \equiv q(m,n) = mn + c$ where m is an integer, $m \geq 1$ and c is integer satisfying $0 \leq c < m$ and $0 \leq n$.

We begin by considering the interpolation at zeros of Chebyshev polynomials. Let

$$T_q(\xi_j^{(q)}) = 0, \qquad j = 1, 2, \ldots, q.$$

Given $f \in A(C_\rho)$ we put

$$a_k^{(q)} = \frac{2}{q}\sum_{i=1}^{q} f(\xi_i^{(q)})T_k(\xi_i^{(q)}), \qquad k = 0, 1, \ldots \tag{7.1.3}$$

that is $a_k^{(q)}$ is the result of approximating A_k (given by (7.1.2)) by the appropriate Gaussian quadrature formula. On substituting (7.1.1) in (7.1.3) and using a property of Chebyshev polynomials (see (18) [39]) we obtain for $k \leq q$,

$$a_k^{(q)} = A_k + \sum_{j=1}^{\infty}(-1)^j(A_{2jq-k} + A_{2jq+k}).$$

Now put

$$u_{n-1,q}(z; f) = {\sum_{k=0}^{n-1}}' a_k^{(q)} T_k(z), \qquad n \leq q.$$

Then it is known [39],

$$u_{n-1,q}(z; f) = S_{n-1}(z; L_{q-1}(f, T))$$

where $S_{n-1}(g)$ is the $n-1^{th}$ partial sum of the chebyshev series of $g(z)$ and $L_{q-1}(f.T)$ is the Lagrange interpolating polynomial of degree at most $q-1$ to f at zeros of $T_q(z)$. Moreover if $q > n-1$, $u_{n-1,q}(z; f)$ is the least squares approximation of degree $n-1$ to f on $\{\xi_1^{(q)}, \ldots, \xi_q^{(q)}\}$ and $u_{n-1,n+1}$ is the best uniform approximation to f by polynomial of degree at most $n-1$ on $\{\xi_1^{(n+1)}, \xi_2^{(n+1)}, \ldots, \xi_{n+1}^{(n+1)}\}$. Thus,

$$u_{n-1,q}(z; f) = {\sum_{k=0}^{n-1}}'\left(A_k + \sum_{j=1}^{\infty}(-1)^j(A_{2jq-k} + A_{2jq+k})\right) T_k(z). \tag{7.1.4}$$

Put

$$s_{n-1\,0}(z; f) = {\sum_{k=0}^{n-1}}' A_k T_k(z). \tag{7.1.5}$$

With these notations Rivlin [39] proved that

Theorem 7.1.1 [39] *If $f \in A(C_\rho)$, $\rho > 1$ and m is an integer greater than 1 then*

$$\lim_{n\to\infty}\{u_{n-1,q}(z; f) - s_{n-1.0}(z; f)\} = 0 \tag{7.1.6}$$

for z inside $C_{\rho^{2m-1}}$, the convergence being uniform and geometric inside and on C_R for any $R < \rho^{2m-1}$, where $q = mn + c$ with $m > 1$ and c a fixed integer satisfying $0 \le c < m$. Moreover, the result is best possible in the sense that (7.1.6) does not hold on $C_{\rho^{2m-1}}$.

Now as in [39] consider extrema of Chebyshev polynomials. Let

$$T_q(\eta_j^{(q)}) = (-1)^j, \qquad j = 0, 1, \dots, q.$$

Given $f \in A(C_\rho)$, put

$$b_k^{(q)} = \frac{2}{q} \sum_{i=0}^{q}{}'' f(\eta_i^{(q)}) T_k(\eta_i^{(q)}), \qquad k = 0, 1, \dots \tag{7.1.7}$$

(double stroke on the summation sign means that the first and last terms are to be halved) that is $b_k^{(q)}$ is the result of approximating A_k (given by (7.1.2)) by the Lobatto Markov quadrature formula. On substituting (7.1.1) in (7.1.7) and using a property of Chebyshev polynomials (see (19) [39]) we obtain for $k \le q$,

$$b_k^{(q)} = A_k + \sum_{j=1}^{\infty} (A_{2jq-k} + A_{2jq+k}).$$

If $n \le q$, put

$$t_{n-1,q}(z; f) = \sum_{k=0}^{n-1}{}' b_k^{(q)} T_k(z),$$

Then it is known [39],

$$t_{n-1,q}(z; f) = S_{n-1}(z; L_q(f, U))$$

where $S_{n-1}(g)$ is the $(n-1)^{th}$ partial sum of the chebyshev series of $g(z)$ and $L_q(f, U)$ is the lagrange interpolating polynomial of degree at most q to f at extremas of $T_q(z)$. Moreover if $q > n-1$, $t_{n-1,q}(z; f)$ is the weighted least squares approximation of degree $n-1$ to f on $\{\eta_0^{(q)}, \dots, \eta_q^{(q)}\}$, the weight 1 being associated with $\eta_i^{(q)}, 0 < i < q$ and weight $\frac{1}{2}$ with $\eta_0^{(q)}, \eta_q^{(q)}$, while $t_{n-1,n}(z; f)$ is the best uniform approximation to f by polynomial of degree at most $n-1$ on $\{\eta_0^{(n)}, \dots, \eta_n^{(n)}\}$. Thus

$$t_{n-1,q}(z; f) = \sum_{k=0}^{n-1}{}' \left(A_k + \sum_{j=1}^{\infty} (A_{2jq-k} + A_{2jq+k}) \right) T_k(z). \tag{7.1.8}$$

With these notations Rivlin [39] proved that

Theorem 7.1.2 [39] *If $f \in A(C_\rho)$, $\rho > 1$ and m is an integer greater than 1 then*

$$\lim_{n \to \infty} \{t_{n-1,q}(z; f) - s_{n-1\,0}(z; f)\} = 0 \tag{7.1.9}$$

for z inside $C_{\rho^{2m-1}}$, the convergence being uniform and geometric inside and on C_R for any $R < \rho^{2m-1}$, where $q = mn + c$ with $m > 1$ and c a fixed integer satisfying $0 \leq c < m$. Moreover, the result is best possible in the sense that (7.1.9) does not hold on $C_{\rho^{2m-1}}$.

Further Rivlin [39] had considered

$$w_{n-1,q}(z;f) = \frac{u_{n-1,q}(z;f) + t_{n-1,q}(z;f)}{2}$$

which is the average of least-squares approximations of f on the chebyshev zeros and extremas. Hence from (7.1.4) and (7.1.8) we have

$$\begin{aligned} w_{n-1,q}(z;f) &= \sum_{k=0}^{n-1}{}' \left(A_k + \sum_{j=1}^{\infty} \frac{(1+(-1)^j}{2}(A_{2jq-k} + A_{2jq+k}) \right) T_k(z) \\ &= \sum_{k=0}^{n-1}{}' \left(A_k + \sum_{j=1}^{\infty}(A_{4jq-k} + A_{4jq+k}) \right) T_k(z). \end{aligned}$$

Put

$$\Gamma_{n-1,1,q}(z;f) = w_{n-1,q}(z;f) - s_{n-1,0}(z;f).$$

Since from (7.1.5)

$$s_{n-1.0}(z) = \sum_{k=0}^{n-1} A_k T_k(z)$$

thus,

$$\Gamma_{n-1,1,q}(z;f) = \sum_{k=0}^{n-1}{}' \left(\sum_{j=1}^{\infty}(A_{4jq-k} + A_{4jq+k}) \right) T_k(z).$$

With the above notations Rivlin [39] proved that

Theorem 7.1.3 [39] *If $f \in A(C_\rho)$, $\rho > 1$, then*

$$\lim_{n\to\infty} (t_{n-1,q}(z;f) - s_{n-1,0}(z;f)) = 0 \tag{7.1.10}$$

for z inside $C_{\rho^{4m-1}}$. The convergence being uniform and geometric inside and on C_R for any $R < \rho^{4m-1}$, where $q = mn + c$ with $m \geq 1$ and c a fixed integer satisfying $0 \leq c < m$. Moreover, the result is best possible in the sense that (7.1.10) does not hold on $C_{\rho^{4m-1}}$.

If for $0 \leq \lambda \leq 1$ we consider

$$W_{n-1,q,\lambda}(z;f) = \lambda u_{n-1,q}(z;f) + (1-\lambda)t_{n-1,q}(z;f)$$

then from (7.1.4), (7.1.5) and (7.1.8)

$$\begin{aligned} W_{n-1,q}(z;f) - s_{n-1,0}(z;f) &= \lambda u_{n-1,q}(z;f) + (1-\lambda)t_{n-1,q}(z;f) - \\ &\quad \lambda s_{n-1,0}(z;f) - (1-\lambda)s_{n-1,0}(z;f) \\ &= \lambda(u_{n-1,q}(z;f) - s_{n-1,0}(z;f)) + \\ &\quad (1-\lambda)(t_{n-1,q}(z;f) - s_{n-1,0}(z;f)) \\ &= \sum_{k=0}^{n-1}{}' \left(\lambda \sum_{j=1}^{\infty} (-1)^j (A_{2jq-k} + A_{2jq+k}) + \right. \\ &\quad \left. + (1-\lambda) \sum_{j=1}^{\infty} (A_{2jq-k} + A_{2jq+k}) \right) T_k(z) \qquad (7.1.11) \\ &= \sum_{k=0}^{n-1}{}' \left(\sum_{j=1}^{\infty} (A_{2jq-k} + A_{2jq+k}) + \right. \\ &\quad \left. - 2\lambda \sum_{j \text{ is odd}} (A_{2jq-k} + A_{2jq+k}) \right) T_k(z) \\ &= \sum_{k=0}^{n-1}{}' \left((1-2\lambda) \sum_{j=1}^{\infty} (A_{2(2j-1)q-k} + A_{2(2j-1)q+k}) \right. \\ &\quad \left. + \sum_{j=1}^{\infty} (A_{4jq-k} + A_{4jq+k}) \right) T_k(z) \end{aligned}$$

Looking at above expression we find that for $\lambda \neq \frac{1}{2}$, $\{W_{n-1,q}(z;f) - s_{n-1,0}(z;f)\}_{n=1}^{\infty}$ will converge to zero in $C_{\rho^{2m-1}}$ and for $\lambda = \frac{1}{2}$, $\{W_{n-1,q}(z;f) - s_{n-1,0}(z;f)\}_{n=1}^{\infty}$ will converge to zero in the larger domain $C_{\rho^{4m-1}}$ as shown in Theorem 7.1.1, Theorem 7.1.2 and Theorem 7.1.3.

Put for $j = 1, 2, \ldots$

$$s_{n-1,j,\lambda}(z) = \sum_{k=0}^{n-1}{}' \Big(\lambda(-1)^j (A_{2jq-k} + A_{2jq+k}) + (1-\lambda)(A_{2jq-k} + A_{2jq+k}) \Big) T_k(z). \qquad (7.1.12)$$

For $l \geq 1$ define

$$\Gamma_{n-1,l,q,\lambda}(z;f) = W_{n-1,q,\lambda}(z;f) - s_{n-1,0}(z;f) - \sum_{j=1}^{l-1} s_{n-1,j,\lambda}(z;f) \qquad (7.1.13)$$

and

$$g_{l,m,\lambda}(R) = \overline{\lim_{n\to\infty}} \max_{z \in C_R} |\Gamma_{n-1,l,q,\lambda}(z;f)|^{1/n} \qquad (7.1.14)$$

Motivated by the results of Cavaretta et al [12], Ivanov & Sharma [19] and Totik [55], in the present chapter, we first extend and then make exact the Rivlin's result, Theorem

7.1.1, Theorem 7.1.2 and Theorem 7.1.3 for the functions analytic inside a ellipse and represented by Chebyshev series. We also consider the poinwise behaviour of the sequence $\{\Gamma_{n-1,l,q,\lambda}(z;f)\}$ inside as well as outside its region of convergence.

7.2 In this section we give convergence result for the sequence $\{\Gamma_{n-1,l,q,\lambda}(z;f)\}$ which for particular cases give Theorem 7.1.1, Theorem 7.1.2 and Theorem 7.1.3. Next we make this result exact. We first have

Theorem 7.2.1 *If $f \in A(C_\rho)$, $\rho > 1$, $l \geq 1$ then for $R > 1$ and $ml > 1$*

$$g_{l,m,\lambda}(R) \leq \frac{R}{\rho^{2lm-1}}, \qquad for \lambda \neq \frac{1}{2} \tag{7.2.1}$$

and for $ml \geq 1$

$$g_{l,m,\lambda}(R) \leq \frac{R}{\rho^{4lm-1}}, \qquad for \lambda = \frac{1}{2}. \tag{7.2.2}$$

More precisely, for $\lambda \neq \frac{1}{2}$,

$$\lim_{n\to\infty} (\Gamma_{n-1,l,q,\lambda}(z;f)) = 0, \tag{7.2.3}$$

for z inside $C_{\rho^{2lm-1}}$. The convergence being uniform and geometric inside and on C_R for any $R < \rho^{2lm-1}$, where $q = mn + c$ with c a fixed integer satisfying $0 \leq c < m$. Moreover, the result is best possible in the sense that (7.2.3) does not hold on $C_{\rho^{2lm-1}}$ and for $\lambda = \frac{1}{2}$

$$\lim_{n\to\infty} \left(\Gamma_{n-1,l,q,\frac{1}{2}}(z;f)\right) = 0 \tag{7.2.4}$$

for z inside $C_{\rho^{4lm-1}}$. The convergence being uniform and geometric inside and on C_R for any $R < \rho^{4lm-1}$, where $q = mn + c$ with c a fixed integer satisfying $0 \leq c < m$. Moreover, the result is best possible in the sense that (7.2.4) does not hold on $C_{\rho^{4lm-1}}$.

Proof As mentioned in the begning since $f \in A(C_\rho)$ we have

$$A_k = \mathcal{O}(\rho - \epsilon)^{-k} \tag{7.2.5}$$

for every ϵ satisfying $0 < \epsilon < \rho - 1$ and $k \geq k_0(\epsilon)$.

Let $z \in C_R, R > 1$ then for $\lambda \neq \frac{1}{2}$ clearly from (7.1.11), (7.1.12) and (7.1.13)

$$\begin{aligned}\Gamma_{n-1,l,q,\lambda}(z;f) &= \sum_{k=0}^{n-1}\sum_{j=l}^{\infty}\Big(\lambda(-1)^j(A_{2jq-k} + A_{2jq+k}) \\ &\quad +(1-\lambda)(A_{2jq-k} + A_{2jq+k})\Big)\, T_k(z)\end{aligned}$$

$$= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} (\lambda_j (A_{2jq-k} + A_{2jq+k}))\, T_k(z), \tag{7.2.6}$$

$$= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} (\lambda_j (A_{2jq-k} + A_{2jq+k}))\, (\frac{w^k + w^{-k}}{2})$$

where $\lambda_j = (\lambda(-1)^j + (1-\lambda)), \qquad j \geq 1$. Hence from (7.2.5)

$$\begin{aligned} |\Gamma_{n-1,l,q,\lambda}(z;f)| &= \mathcal{O}\left(\sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} \left(\frac{1}{(\rho-\epsilon)^{2jq-k}} + \frac{1}{(\rho-\epsilon)^{2jq+k}}\right)\left(\frac{R^k + R^{-k}}{2}\right)\right) \\ &= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{2lq}} \sum_{k=0}^{n-1}\left[((\rho-\epsilon)R)^k + \left(\frac{(\rho-\epsilon)}{R}\right)^k + \right.\right. \\ &\quad \left.\left. + \left(\frac{R}{(\rho-\epsilon)}\right)^k + \left(\frac{1}{(\rho-\epsilon)R}\right)^k\right]\right) \\ &= \mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{2lq-n}}\right) \end{aligned}$$

which gives

$$g_{l,m,\lambda}(R) \leq \frac{R}{(\rho-\epsilon)^{2lm-1}}.$$

ϵ being arbitrary small, we have

$$g_{l,m,\lambda}(R) \leq \frac{R}{\rho^{2lm-1}}.$$

Next for $\lambda = \frac{1}{2}$ from (7.2.6)

$$\begin{aligned} \Gamma_{n-1,l,q,\frac{1}{2}}(z;f) &= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} \left((\frac{1}{2}(-1)^j + (1-\frac{1}{2}))(A_{2jq-k} + \right. \\ &\quad \left. A_{2jq+k})\right)\, T_k(z) \\ &= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} (A_{4jq-k} + A_{4jq+k}) T_k(z) \qquad (7.2.7) \\ &= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} (A_{4jq-k} + A_{4jq+k})(\frac{w^k + w^{-k}}{2}) \end{aligned}$$

hence from (7.2.5)

$$\begin{aligned} |\Gamma_{n-1,l,q,\frac{1}{2}}(z;f)| &= \mathcal{O}\left(\sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} \left(\frac{1}{(\rho-\epsilon)^{4jq-k}} + \frac{1}{(\rho-\epsilon)^{4jq+k}}\right)\left(\frac{R^k + R^{-k}}{2}\right)\right) \\ &= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{4lq}} \sum_{k=0}^{n-1}{}'\left[((\rho-\epsilon)R)^k + \left(\frac{(\rho-\epsilon)}{R}\right)^k + \right.\right. \\ &\quad \left.\left. + \left(\frac{R}{(\rho-\epsilon)}\right)^k + \left(\frac{1}{(\rho-\epsilon)R}\right)^k\right]\right) \\ &= \mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{4lq-n}}\right) \end{aligned}$$

which gives

$$g_{l,m,\frac{1}{2}}(R) \le \frac{R}{(\rho-\epsilon)^{4lm-1}}.$$

ϵ being arbitrary small, we have

$$g_{l,m,\frac{1}{2}}(R) \le \frac{R}{\rho^{4lm-1}}.$$

Now to show that result of (7.2.3) is best possible, consider

$$f_0(z) = \sum_{k=0}^{\infty} \alpha^k T_k(z) \tag{7.2.8}$$

where $0 < \alpha = \rho^{-1} < 1$. Clearly $f_0(z) \in A(C_\rho)$. Put $z_0 = \frac{\rho^{2lm-1}+\rho^{-(2lm-1)}}{2} \in C_{\rho^{2lm-1}}$ then from (7.2.6)

$$\begin{aligned}
\Gamma_{n-1,l,q,\lambda}(z_0; f_0) &= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} (\lambda_j(A_{2jq-k} + A_{2jq+k}))\, T_k(z_0) \\
&= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} \lambda_j(A_{2jq-k} + A_{2jq+k}) T_k(z_0) \\
&= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} \lambda_j(\alpha^{2jq-k} + \alpha^{2jq+k})(\frac{\alpha^{(2ml-1)k} + \alpha^{-(2ml-1)k}}{2}) \\
&= \frac{1}{2}\sum_{j=l}^{\infty} \lambda_j \alpha^{2jq} \sum_{k=0}^{n-1}{}' (\alpha^k + \alpha^{-k})(\alpha^{(2ml-1)k} + \alpha^{-(2ml-1)k})
\end{aligned}$$

hence

$$\begin{aligned}
|\Gamma_{n-1,l,q,\lambda}(z_0; f_0)| &\ge \frac{\alpha^{2ql}}{2(1-\alpha^{2q})} \sum_{k=0}^{n-1}{}' (\alpha^k + \alpha^{-k})(\alpha^{(2ml-1)k} + \alpha^{-(2ml-1)k}) \\
&> \frac{\alpha^{2ql}}{2(1+\alpha^{2q})} \alpha^{-2ml(n-1)} \\
&> \frac{\alpha^{2l(mn+c)}}{4} \alpha^{-2mln+2ml} \\
&> \frac{\alpha^{2lc+2ml}}{4} > 0
\end{aligned}$$

showing that (7.2.3) of Theorem 7.2.1 is not valid at a point on $C_{\rho^{2lm-1}}$ in this case. Now to show that result of (7.2.4) is best possible, consider $z_1 = \frac{\rho^{4ml-1}+\rho^{-(4ml-1)}}{2} \in C_{\rho^{4ml-1}}$ then

$$\begin{aligned}
\Gamma_{n-1,l,q,\frac{1}{2}}(z_1; f_0) &= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} (A_{4jq-k} + A_{4jq+k}) T_k(z_1) \\
&= \sum_{k=0}^{n-1} \sum_{j=l}^{\infty} (\alpha^{4jq-k} + \alpha^{4jq+k})(\frac{\alpha^{(4ml-1)k} + \alpha^{-(4ml-1)k}}{2})
\end{aligned}$$

$$= \frac{1}{2}\sum_{j=l}^{\infty}\alpha^{4jq}\sum_{k=0}^{n-1}{}'(\alpha^k+\alpha^{-k})(\alpha^{(4ml-1)k}+\alpha^{-(4ml-1)k})$$

$$= \frac{1}{2}\frac{\alpha^{4ql}}{1-\alpha^{4q}}\sum_{k=0}^{n-1}{}'(\alpha^k+\alpha^{-k})(\alpha^{(4ml-1)k}+\alpha^{-(4ml-1)k})$$

hence

$$\begin{aligned}|\Gamma_{n-1,l,q,\frac{1}{2}}(z_1;f_0)| &> \frac{\alpha^{4ql}}{2(1+\alpha^{4q})}\alpha^{-4ml(n-1)} \\ &> \frac{\alpha^{4l(mn+c)}}{4}\alpha^{-4mln+4ml} \\ &> \frac{\alpha^{4lc+4ml}}{4} > 0\end{aligned}$$

showing that (7.2.4) of Theorem 7.2.1 is not valid at a point on $C_{\rho^{4ml-1}}$. This completes the proof of the theorem.

Remark 7.2.1 For $\lambda = 1$ and $l = 1$ Theorem 7.2.1 reduces to Theorem 7.1.1.

Remark 7.2.2 For $\lambda = 0$ and $l = 1$ Theorem 7.2.1 reduces to Theorem 7.1.2.

Remark 7.2.3 For $\lambda = \frac{1}{2}$ and $l = 1$ Theorem 7.2.1 reduces to Theorem 7.1.3.

Remark 7.2.4 For the case $\lambda \neq \frac{1}{2}$ from (7.2.6) we have

$$\Gamma_{n-1,l,q,\lambda}(z;f) = \sum_{k=0}^{n-1}{}'\sum_{j=l}^{\infty}(\lambda_j(A_{2jq-k}+A_{2jq+k}))\,T_k(z).$$

Hence for $q = n$ i.e $m = 1, c = 0$ and $l = 1$ we have

$$\Gamma_{n-1,1,n,\lambda}(z;f) = \sum_{k=0}^{n-1}{}'\sum_{j=1}^{\infty}(\lambda_j(A_{2jn-k}+A_{2jn+k}))\,T_k(z).$$

Now consider $f_0(z)$ given by (7.2.8) and put $z_2 = \frac{\rho+\rho^{-1}}{2} \in C_\rho$ then

$$\begin{aligned}\Gamma_{n-1,1,n,\lambda}(z_2;f_0) &= \sum_{k=0}^{n-1}{}'\sum_{j=1}^{\infty}(\lambda_j(A_{2jn-k}+A_{2jn+k}))\,T_k(z_2) \\ &= \sum_{k=0}^{n-1}{}'\sum_{j=1}^{\infty}\lambda_j(\alpha^{2jn-k}+\alpha^{2jn+k})(\frac{\alpha^k+\alpha^{-k}}{2}) \\ &= \frac{1}{2}\sum_{j=1}^{\infty}\lambda_j\alpha^{2jn}\sum_{k=0}^{n-1}{}'(\alpha^k+\alpha^{-k})(\alpha^k+\alpha^k)\end{aligned}$$

hence

$$|\Gamma_{n-1,1,n,\lambda}(z_2;f_0)| \geq \frac{\alpha^{2n}}{2(1-\alpha^{2n})}\sum_{k=0}^{n-1}{}'(\alpha^k+\alpha^{-k})^2$$

$$
\begin{aligned}
&> \frac{\alpha^{2nl}}{2(1+\alpha^{2n})}\alpha^{-2(n-1)} \\
&> \frac{\alpha^{2n}}{4}\alpha^{-2n+2} \\
&> \frac{\alpha^{2}}{4} > 0.
\end{aligned}
$$

Which shows that for $\lambda \neq \frac{1}{2}$, $\Gamma_{n-1,1,n,\lambda}(z;f)$ does not exhibit Walsh equiconvergence for $f = f_0$ and $z = z_2$, while for $\lambda = \frac{1}{2}$ for $q = n$ i.e. $m = 1, c = 0$ and $l = 1$ from Theorem 7.2.1, $\Gamma_{n-1,1,n,\frac{1}{2}}(z;f)$ does have Walsh equiconvergence property within C_{ρ^3}.

Next, we make exact Theorem 7.2.1. In fact we show that in (7.2.1) and (7.2.2) equality holds always.

Theorem 7.2.2 *If $f \in A(C_\rho), \rho > 1, l$ is a positive integer and $R > 1$ then for $ml > 1$*

$$g_{l,m,\lambda}(R) = \frac{R}{\rho^{2ml-1}}, \qquad for \lambda \neq \frac{1}{2}$$

and for $ml \geq 1$

$$g_{l,m,\lambda}(R) = \frac{R}{\rho^{4ml-1}} \qquad for \lambda = \frac{1}{2}.$$

Proof Let R be fixed, $|w| = R$ and $1 < R$. Then for $\lambda \neq \frac{1}{2}$ from (7.2.1) we have

$$g_{l,m,\lambda}(R) \leq \frac{R}{\rho^{2ml-1}} \qquad \text{for} \qquad 1 < R < \infty$$

To prove the opposite inequality, from (7.2.6) we have,

$$
\begin{aligned}
\Gamma_{n-1,l,q,\lambda}(z;f) &= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} (\lambda_j(A_{2jq-k} + A_{2jq+k}))\, T_k(z) \\
&= \sum_{k=0}^{n-1}{}' (\lambda_l(A_{2lq-k} + A_{2lq+k}))\, T_k(z) + \\
&\quad \sum_{k=0}^{n-1}{}' \sum_{j=l+1}^{\infty} (\lambda_j(A_{2jq-k} + A_{2jq+k}))\, T_k(z) \\
&= \sum_{k=0}^{n-2ml}{}' \lambda_l(A_{2lq-k} + A_{2lq+k}) \left(\frac{w^k + w^{-k}}{2}\right) + \\
&\quad + \sum_{k=n-2ml+1}^{n-1} \lambda_l(A_{2lq-k} + A_{2lq+k}) \left(\frac{w^k + w^{-k}}{2}\right) + \\
&\quad + \sum_{k=0}^{n-1}{}' \sum_{j=l+1}^{\infty} \lambda_j(A_{2jq-k} + A_{2jq+k}) \left(\frac{w^k + w^{-k}}{2}\right).
\end{aligned}
$$

Thus,

$$\begin{aligned}
\sum_{k=n-2ml+1}^{n-1} \frac{1}{2}\lambda_l A_{2lq-k} w^k &= \Gamma_{n-1,l,q,\lambda}(z;f) - \sum_{k=0}^{n-2ml}{}' \lambda_l (A_{2lq-k} + A_{2lq+k}) \left(\frac{w^k + w^{-k}}{2}\right) \\
&\quad - \sum_{k=n-2ml+1}^{n-1} \lambda_l \left(A_{2lq+k} \left(\frac{w^k + w^{-k}}{2}\right) + A_{2lq-k} \frac{w^{-k}}{2}\right) \\
&\quad - \sum_{k=0}^{n-1}{}' \sum_{j=l+1}^{\infty} \lambda_j (A_{2jq-k} + A_{2jq+k}) \left(\frac{w^k + w^{-k}}{2}\right)
\end{aligned}$$

gives, by Cauchy integral formula, for $n - 2ml + 1 \le k \le n - 1$,

$$\begin{aligned}
\frac{1}{2}\lambda_l A_{2lq-k} &= \frac{1}{2\pi i} \int_{|w|=R} \frac{\Gamma_{n-1,l,q,\lambda}(z;f)}{w^{k+1}} dw - \\
&\quad - \frac{1}{2\pi i} \sum_{k'=0}^{n-2ml}{}' \frac{1}{2}\lambda_l (A_{2lq-k'} + A_{2lq+k'}) \int_{|w|=R} \frac{w^{k'} + w^{-k'}}{w^{k+1}} dw - \\
&\quad - \frac{1}{2\pi i} \sum_{k'=n-2ml+1}^{n-1} \frac{1}{2}\lambda_l A_{2lq+k'} \int_{|w|=R} \frac{w^{k'} + w^{-k'}}{w^{k+1}} dw - \\
&\quad - \frac{1}{2\pi i} \sum_{k'=n-2ml+1}^{n-1} \frac{1}{2}\lambda_l A_{2lq-k'} \int_{|w|=R} \frac{w^{-k'}}{w^{k+1}} dw - \\
&\quad - \frac{1}{2\pi i} \int_{|w|=R} \frac{\sum_{k'=0}^{\prime n-1} \sum_{j=l+1}^{\infty} \lambda_j (A_{2jq-k'} + A_{2jq+k'}) \left(\frac{w^{k'} + w^{-k'}}{2}\right)}{w^{k+1}} dw \\
&= \frac{1}{2\pi i} \int_{|w|=R} \frac{\Gamma_{n-1,l,q,\lambda}(z;f)}{w^{k+1}} dw - 0 - \frac{1}{2}\lambda_l A_{2lq+k} - \\
&\quad - \frac{1}{2\pi i} \int_{|w|=R} \frac{\sum_{k'=0}^{\prime n-1} \sum_{j=l+1}^{\infty} \lambda_j (A_{2jq-k'} + A_{2jq-k'}) \left(\frac{w^{k'} + w^{-k'}}{2}\right)}{w^{k+1}} dw.
\end{aligned}$$

Hence by the definition of $g_{l,m,\lambda}(R)$ and (7.2.5) we have for $R > 1$, every $n \ge n_0(\epsilon)$ and a constant M,

$$\begin{aligned}
|A_{2ql-k}| &\le M \frac{(g_{l,m,\lambda}(R) + \epsilon)^n}{R^k} + \mathcal{O}\left(\frac{1}{(\rho - \epsilon)^{2lq+k}} + \frac{R^{n-k}}{(\rho - \epsilon)^{2(l+1)q-n}}\right) \\
&\le M \frac{(g_{l,m,\lambda}(R) + \epsilon)^n}{R^k} + \mathcal{O}\left(\frac{1}{(\rho - \epsilon)^{2lmn+n}} + \frac{1}{(\rho - \epsilon)^{2(l+1)mn-n}}\right). \qquad (7.2.9)
\end{aligned}$$

Now choose $\epsilon > 0$ so small that $1 < \rho - \epsilon$ and

$$\frac{1}{(\rho - \epsilon)^{2lm+1}} < \frac{1}{\rho^{2lm-1}}$$

and

$$\frac{1}{(\rho - \epsilon)^{2(l+1)m-1}} < \frac{1}{\rho^{2lm-1}}$$

this together with (7.2.9) gives

$$|A_{2ql-k}| \le M \frac{(g_{l,m,\lambda}(R) + \epsilon)^n}{R^k} + \mathcal{O}\left(\frac{1}{\rho^{(2lm-1)n}}\right)$$

or,

$$(g_{l,m,\lambda}(R)+\epsilon)^n \geq \frac{R^k}{M}\left(|A_{2ql-k}| - O\left(\frac{1}{\rho^{(2lm-1)n}}\right)\right)$$

hence,

$$g_{l,m,\lambda}(R)+\epsilon \geq \varlimsup_{n\to\infty}\left\{\frac{R^k}{M}\right\}^{\frac{1}{n}}\left\{|A_{2ql-k}|^{\frac{1}{2ql-k}}\right\}^{\frac{2ql-k}{n}}.$$

Now since $n-2ml+1 \leq k \leq n-1$ we have, $\lim_{n\to\infty}\frac{k}{n}=1$ and so

$$g_{l,m,\lambda}(R)+\epsilon \geq \frac{R}{\rho^{2lm-1}}.$$

Since ϵ is arbitrary, this yields

$$g_{l,m,\lambda}(R) \geq \frac{R}{\rho^{2lm-1}}, \qquad \text{for } \; R>1$$

which completes the proof for $\lambda \neq \frac{1}{2}$.

Next for $\lambda=\frac{1}{2}$ from (7.2.2) we have

$$g_{l,m,\lambda}(R) \leq \frac{R}{\rho^{4ml-1}}. \qquad \text{for} \qquad 1<R<\infty$$

To prove the opposite inequality, from (7.2.7) we have,

$$\begin{aligned}
\Gamma_{n-1,l,q,\lambda}(z;f) &= \sum_{k=0}^{n-1}{}'\sum_{j=l}^{\infty}(A_{4jq-k}+A_{4jq+k})T_k(z) \\
&= \sum_{k=0}^{n-1}{}'(A_{4lq-k}+A_{4lq+k})\left(\frac{w^k+w^{-k}}{2}\right)+ \\
&\quad +\sum_{k=0}^{n-1}{}'\sum_{j=l+1}^{\infty}(A_{4jq-k}+A_{4jq+k})\left(\frac{w^k+w^{-k}}{2}\right) \\
&= \sum_{k=0}^{n-4ml}{}'(A_{4lq-k}+A_{4lq+k})\left(\frac{w^k+w^{-k}}{2}\right)+ \\
&\quad +\sum_{k=n-4ml+1}^{n-1}(A_{4lq-k}+A_{4lq+k})\left(\frac{w^k+w^{-k}}{2}\right)+ \\
&\quad +\sum_{k=0}^{n-1}{}'\sum_{j=l+1}^{\infty}(A_{4jq-k}+A_{4jq+k})\left(\frac{w^k+w^{-k}}{2}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=n-4ml+1}^{n-1}\frac{1}{2}A_{4lq-k}w^k &= \Gamma_{n-1,l,q,\frac{1}{2}}(z;f)-\sum_{k=0}^{n-4ml}{}'(A_{4lq-k}+A_{4lq+k})\left(\frac{w^k+w^{-k}}{2}\right) \\
&\quad -\sum_{k=n-4ml+1}^{n-1}\left(A_{4lq+k}\left(\frac{w^k+w^{-k}}{2}\right)+A_{4lq-k}\frac{w^{-k}}{2}\right) \\
&\quad -\sum_{k=0}^{n-1}{}'\sum_{j=l+1}^{\infty}(A_{4jq-k}+A_{4jq+k})\left(\frac{w^k+w^{-k}}{2}\right)
\end{aligned}$$

gives, by Cauchy integral formula, for $n-4ml+1 \leq k \leq n-1$,

$$\begin{aligned}
\frac{1}{2}A_{4lq-k} &= \frac{1}{2\pi i}\int_{|w|=R}\frac{\Gamma_{n-1,l,q,\frac{1}{2}}(z;f)}{w^{k+1}}dw - \\
&-\frac{1}{2\pi i}\sum_{k'=0}^{n-4ml}{}' \frac{1}{2}(A_{4lq-k'}+A_{4lq+k'})\int_{|w|=R}\frac{w^{k'}+w^{-k'}}{w^{k+1}}dw - \\
&-\frac{1}{2\pi i}\sum_{k'=n-4ml+1}^{n-1}\frac{1}{2}A_{4lq+k'}\int_{|w|=R}\frac{w^{k'}+w^{-k'}}{w^{k+1}}dw - \\
&-\frac{1}{2\pi i}\sum_{k'=n-4ml+1}^{n-1}\frac{1}{2}A_{4lq-k'}\int_{|w|=R}\frac{w^{-k'}}{w^{k+1}}dw - \\
&-\frac{1}{2\pi i}\int_{|w|=R}\frac{\sum_{k'=0}^{n-1}\sum_{j=l+1}^{\infty}(A_{4jq-k'}+A_{4jq+k'})\left(\frac{w^{k'}+w^{-k'}}{2}\right)}{w^{k+1}}dw \\
&= \frac{1}{2\pi i}\int_{|w|=R}\frac{\Gamma_{n-1,l,q,\frac{1}{2}}(z;f)}{w^{k+1}}dw - 0 - \frac{1}{2}A_{4lq+k} - \\
&-\frac{1}{2\pi i}\int_{|w|=R}\frac{\sum_{k'=0}^{n-1}\sum_{j=l+1}^{\infty}(A_{4jq-k'}+A_{4jq+k'})\left(\frac{w^{k'}+w^{-k'}}{2}\right)}{w^{k+1}}dw.
\end{aligned}$$

Hence by the definition of $g_{l,m,\frac{1}{2}}(R)$ and (7.2.5) we have for $R>1$, every $n \geq n_0(\epsilon)$ and a constant M,

$$\begin{aligned}
|A_{4ql-k}| &\leq M\frac{(g_{l,m,\frac{1}{2}}(R)+\epsilon)^n}{R^k}+\mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{4lq+k}}+\frac{R^{n-k}}{(\rho-\epsilon)^{4(l+1)q-n}}\right) \\
&\leq M\frac{(g_{l,m,\frac{1}{2}}(R)+\epsilon)^n}{R^k}+\mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{4lmn+n}}+\frac{1}{(\rho-\epsilon)^{4(l+1)mn-n}}\right) \qquad (7.2.10)
\end{aligned}$$

Now choose $\epsilon>0$ so small that $1<\rho-\epsilon$ and

$$\frac{1}{(\rho-\epsilon)^{4lm+1}}<\frac{1}{\rho^{4lm-1}}$$

and

$$\frac{1}{(\rho-\epsilon)^{4(l+1)m-1}}<\frac{1}{\rho^{4lm-1}}$$

this together with (7.2.10) gives

$$|A_{4ql-k}| \leq M\frac{(g_{l,m,\frac{1}{2}}(R)+\epsilon)^n}{R^k}+\mathcal{O}\left(\frac{1}{\rho^{(4lm-1)n}}\right)$$

or,

$$(g_{l,m,\frac{1}{2}}(R)+\epsilon)^n \geq \frac{R^k}{M}\left(|A_{4ql-k}|-\mathcal{O}\left(\frac{1}{\rho^{(4lm-1)n}}\right)\right)$$

hence,

$$g_{l,m,\frac{1}{2}}(R)+\epsilon \geq \overline{\lim_{n\to\infty}}\left\{\frac{R^k}{M}\right\}^{\frac{1}{n}}\left\{|A_{4ql-k}|^{\frac{1}{4ql-k}}\right\}^{\frac{4ql-k}{n}}.$$

Now since $n-4ml+1 \le k \le n-1$ we have, $\lim_{n\to\infty} \frac{k}{n} = 1$ and so

$$g_{l,m,\frac{1}{2}}(R) + \epsilon \ge \frac{R}{\rho^{4lm-1}}.$$

Since ϵ is arbitrary, this yields

$$g_{l,m,\frac{1}{2}}(R) \ge \frac{R}{\rho^{4lm-1}}, \qquad \text{for} \quad R > 1$$

which completes the proof for $\lambda = \frac{1}{2}$.

7.3 Our next concern will be the pointwise behaviour of $\{\Gamma_{n-1,l,q,\lambda}(z;f)\}$. We shall not only prove that $\{\Gamma_{n-1,l,q,\lambda}(z;f)\}$ is bounded at most at some finite number of points outside its region of convergence but

Theorem 7.3.1 *Let $f \in A(C_\rho), \rho > 1$ and $l \ge 1$. Then for $ml > 1$*

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\lambda}(z:f)|^{1/n} = \frac{R}{\rho^{2ml-1}}, \qquad z \in C_R, R > 1, \ for \lambda \ne \frac{1}{2} \tag{7.3.1}$$

for all but at most $2lm-2$ points outside $[-1,1]$ and for $ml \ge 1$

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\lambda}(z:f)|^{1/n} = \frac{R}{\rho^{4ml-1}}, \qquad z \in C_R, R > 1 \ for \lambda = \frac{1}{2} \tag{7.3.2}$$

for all but at most $4lm-2$ points outside $[-1,1]$.

Proof For $\lambda \ne \frac{1}{2}$, consider,

$$h_\lambda(z) = \Gamma_{n-1,l,q,\lambda}(z;f) - w^{-2ml}\Gamma_{n,l,q,\lambda}(z;f),$$

where $z = (w+w^{-1})/2$. Hence from (7.2.6) we have

$$\begin{aligned}
h_\lambda(z) &= \Gamma_{n-1,l,q,\lambda}(z;f) - w^{-2ml}\Gamma_{n,l,q,\lambda}(z;f) \\
&= \sum_{k=0}^{n-1}{}' \sum_{j=l}^{\infty} \lambda_j (A_{2jq-k} + A_{2jq+k})(\frac{w^k + w^{-k}}{2}) - \\
&\quad \sum_{k=0}^{n}{}' \sum_{j=l}^{\infty} (A_{4jq+4mj-k} + A_{2jq+2mj+k})(\frac{w^k + w^{-k}}{2}) w^{-2ml} \\
&= \sum_{k=0}^{n-1}{}' \lambda_l (A_{2lq-k} + A_{2lq+k})(\frac{w^k + w^{-k}}{2}) - \\
&\quad \sum_{k=0}^{n}{}' \lambda_l (A_{2lq+2ml-k} + A_{2lq+2ml+k})(\frac{w^{k-2ml} + w^{-k-2ml}}{2}) + \\
&\quad + \Bigg[\sum_{k=0}^{n-1}{}' \sum_{j=l+1}^{\infty} \lambda_j (A_{2jq-k} + A_{2jq+k})(\frac{w^k + w^{-k}}{2}) -
\end{aligned}$$

$$\sum_{k=0}^{n}\sum_{j=l+1}^{\infty}\lambda_j(A_{2jq+2mj-k}+A_{2jq+2mj+k})(\frac{w^k+w^{-k}}{2})w^{-2ml}\Bigg]$$

$$\begin{aligned}
&= \left(\sum_{k=0}^{n-1}\frac{1}{2}\lambda_l A_{2lq-k}w^k-\sum_{k=0}^{n}\frac{1}{2}\lambda_l A_{2lq+2ml-k}w^{k-2ml}\right)+ \\
&\quad +\sum_{k=0}^{n-1}\frac{1}{2}\lambda_l A_{2lq-k}w^{-k}+\sum_{k=0}^{n-1}\frac{1}{2}\lambda_l A_{2lq+k}(w^k+w^{-k}) \\
&\quad -\sum_{k=0}^{n}\frac{1}{2}\lambda_l A_{2lq+2ml-k}w^{-k-2ml} \\
&\quad -\sum_{k=0}^{n}\frac{1}{2}\lambda_l A_{2lq+2ml+k}(w^{k-2ml}+w^{-k-2ml})+\Theta \qquad (7.3.3)
\end{aligned}$$

where

$$\begin{aligned}
\Theta &= \Bigg[\sum_{k=0}^{n-1}{}'\sum_{j=l+1}^{\infty}\lambda_j(A_{2jq-k}+A_{2jq+k})(\frac{w^k+w^{-k}}{2})- \\
&\sum_{k=0}^{n}{}'\sum_{j=l+1}^{\infty}\lambda_j(A_{2jq+2mj-k}+A_{2jq+2mj+k})(\frac{w^k+w^{-k}}{2})w^{-2ml}\Bigg].
\end{aligned}$$

Note that for $R>1$,

$$\Theta=\mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{(2m(l+1)-1)n}}\right) \qquad (7.3.4)$$

Let $R\geq\rho$, thus from (7.3.3) and (7.3.4) we have

$$\begin{aligned}
h_\lambda(z) &= \frac{1}{2}\left(\sum_{k=0}^{n-1}{}'-\sum_{k=-2ml}^{n-2ml}{}'\right)\lambda_l A_{2lq-k}w^k+\mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{2lq}}+\frac{R^n}{(\rho-\epsilon)^{2lq+n}}+\right. \\
&\quad \left.+\frac{1}{(\rho-\epsilon)^{2lq}}+\frac{R^n}{(\rho-\epsilon)^{2lq+n}}\right)+\mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{(2m(l+1)-1)n}}\right) \\
&= \frac{1}{2}\left(\sum_{k=n+1-2ml}^{n-1}-\sum_{k=-2ml}^{-1}{}'\right)\lambda_l A_{2lq-k}w^k+ \\
&\quad +\mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{2lmn+n}}+\frac{R^n}{(\rho-\epsilon)^{(2m(l+1)-1)n}}\right) \\
&= \frac{1}{2}\sum_{k=0}^{2ml-2}\lambda_l A_{2lq-k-(n+1-2ml)}w^{k+n+1-2ml}+\mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{2lmn+n}}\right). \qquad (7.3.5)
\end{aligned}$$

Note that for $R\geq\rho$, by choosing ϵ sufficiently small we can find $\eta_1>0$ such that

$$\frac{R^n}{(\rho-\epsilon)^{2lmn+n}}<\left(\frac{R}{\rho^{2lm-1}}-\eta_1\right)^n. \qquad (7.3.6)$$

Similarly for $1<R\leq\rho$ from (7.3.3) and (7.3.4) we have

$$h_\lambda(z) = \frac{1}{2}\left(\sum_{k=0}^{n-1}{}'-\sum_{k=-2ml}^{n-2ml}{}'\right)\lambda_l A_{2lq-k}w^k+\mathcal{O}\left(\frac{R^{-n}}{(\rho-\epsilon)^{2lq-n}}+\frac{1}{(\rho-\epsilon)^{2lq}}+\right.$$

$$+\frac{R^{-n}}{(\rho-\epsilon)^{2lq-n}}+\frac{1}{(\rho-\epsilon)^{2lq}}\Bigg)+\mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{(2m(l+1)-1)n}}\right)$$

$$=\frac{1}{2}\left(\sum_{k=n+1-2ml}^{n-1}-\sum_{k=-2ml}^{-1}{}'\right)\lambda_l A_{2lq-k}w^k+$$

$$\mathcal{O}\left(\frac{R^{-n}}{(\rho-\epsilon)^{2lmn-n}}+\frac{R^n}{(\rho-\epsilon)^{(2m(l+1)-1)n}}\right)$$

$$=\frac{1}{2}\sum_{k=0}^{2ml-2}\lambda_l A_{2lq-k-(n+1-2ml)}w^{k+n+1-2ml}+\mathcal{O}\left(\frac{R^{-n}}{(\rho-\epsilon)^{2lmn-n}}\right). \tag{7.3.7}$$

Again for $1<R\le\rho$, by choosing ϵ sufficiently small we can find $\eta_2>0$ such that

$$\frac{R^{-n}}{(\rho-\epsilon)^{2lmn-n}}<\left(\frac{R}{\rho^{2lm-1}}-\eta_2\right)^n. \tag{7.3.8}$$

Thus from (7.3.5), (7.3.6), (7.3.7) and (7.3.8) we have,

$$h_\lambda(z)=\frac{1}{2}\sum_{k=0}^{2ml-2}\lambda_l A_{2lq-k-(n+1-2ml)}w^{k+n+1-2ml}+\mathcal{O}\left(\frac{R}{\rho^{2lm-1}}-\eta\right)^n \tag{7.3.9}$$

where η is a positive number.

If we assume that in (7.3.1) equality does not hold at more than $2ml-2$ points, say, $2ml-1$ points, that is

$$\overline{\lim_{n\to\infty}}|\Gamma_{n-1,l,q,\lambda}(z_j:f)|^{1/n}<\frac{|w_j|}{\rho^{2ml-1}},\qquad j=1,2,\ldots,2ml-1$$

for $z_1,\ldots,z_{2ml-1}$ with images $|w_1|,\ldots,|w_{2ml-1}|>1$ then we have

$$\overline{\lim_{n\to\infty}}|h_\lambda(z_j)|^{1/n}<\frac{|w_j|}{\rho^{2ml-1}},\qquad j=1,2,\ldots,2ml-1$$

and hence from (7.3.9)

$$\sum_{k=0}^{2ml-2}\lambda_l A_{2lq-k-(n+1-2ml)}w_j^{k+n+1-2ml}=\beta_{j,n},\qquad j=1,\ldots,2ml-1$$

where

$$|\beta_{j,n}|<K_1\left(\frac{|w_j|}{\rho^{2ml-1}}-\eta_1\right)^n$$

for some $\eta_1>0, K_1\ge 1, 1\le j\le 2ml-1$ and $n=1,2,\ldots$. Thus

$$\sum_{k=0}^{2ml-2}A_{2lq-k-(n+1-2ml)}w_j^k=w_j^{-n+2ml-1}\beta_{j,n}.\qquad j=1,\ldots,2ml-1$$

Solving this system of equations for $A_{2lq-k-(n+1-2ml)}$, $k=0,1,\ldots,2ml-2$, we have

$$A_{2lq-k-(n+1-2ml)}=\sum_{j=1}^{2ml-1}c_j^{(k)}w_j^{-n+2ml-1}\beta_{j,n}$$

where $c_j^{(k)}$ are constants independent of n, which gives

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} |A_{2lq-k-(n+1-2ml)}|^{\frac{1}{2lq-k-(n+1-2ml)}} \\
&\leq \overline{\lim_{n\to\infty}} \left(K_2 |w_j|^{-n} \left(\frac{|w_j|}{\rho^{2ml-1}} - \eta_1 \right)^n \right)^{\frac{1}{2lq-k-(n+1-2ml)}} \\
&\leq \overline{\lim_{n\to\infty}} \left(K_2 \left(\frac{1}{\rho^{2ml-1}} - \frac{\eta_1}{max|w_j|} \right)^n \right)^{\frac{1}{2lq-k-(n+1-2ml)}} \\
&< \frac{1}{\rho}
\end{aligned}$$

which is a contradiction to $f \in A(C_\rho)$. Hence our assumption that equality in (7.3.1) does not hold at more than $2ml-2$ points was wrong. Thus

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\lambda}(z:f)|^{1/n} = \frac{R}{\rho^{2ml-1}}, \qquad z \in C_R, R > 1 \ for \lambda \neq \frac{1}{2}$$

for all but at most $2lm-2$ points outside $[-1,1]$.

Next for $\lambda = \frac{1}{2}$ consider,

$$h(z) = \Gamma_{n-1,l,q,\lambda}(z;f) - w^{-4ml}\Gamma_{l,q,n}(z;f)$$

where $z = (w+w^{-1})/2$. Hence from (7.2.7) we have

$$\begin{aligned}
h(z) &= \Gamma_{n-1,l,q,\lambda}(z;f) - w^{-4ml}\Gamma_{l,q,n}(z;f) \\
&= \sum_{k=0}^{n-1}\sum_{j=l}^{\infty}(A_{4jq-k} + A_{4jq+k})(\frac{w^k+w^{-k}}{2}) - \\
&\quad \sum_{k=0}^{n}\sum_{j=l}^{\infty}(A_{4jq+4mj-k} + A_{4jq+4mj+k})(\frac{w^k+w^{-k}}{2})w^{-4ml} \\
&= \sum_{k=0}^{n-1}{}'(A_{4lq-k} + A_{4lq+k})(\frac{w^k+w^{-k}}{2}) - \\
&\quad \sum_{k=0}^{n}{}'(A_{4lq+4ml-k} + A_{4lq+4ml+k})(\frac{w^{k-4ml}+w^{-k-4ml}}{2}) + \\
&\quad + \Bigg[\sum_{k=0}^{n-1}{}' \sum_{j=l+1}^{\infty} (A_{4jq-k} + A_{4jq+k})(\frac{w^k+w^{-k}}{2}) - \\
&\quad \sum_{k=0}^{n}{}' \sum_{j=l+1}^{\infty} (A_{4jq+4mj-k} + A_{4jq+4mj+k})(\frac{w^k+w^{-k}}{2})w^{-4ml} \Bigg] \\
&= \left(\sum_{k=0}^{n-1}{}' \frac{1}{2} A_{4lq-k} w^k - \sum_{k=0}^{n}{}' \frac{1}{2} A_{4lq+4ml-k} w^{k-4ml} \right) + \\
&\quad + \sum_{k=0}^{n-1}{}' \frac{1}{2} A_{4lq-k} w^{-k} + \sum_{k=0}^{n-1}{}' \frac{1}{2} A_{4lq+k}(w^k + w^{-k})
\end{aligned}$$

$$-\sum_{k=0}^{n}{}' \frac{1}{2} A_{4lq+4ml-k} w^{-k-4ml}$$
$$-\sum_{k=0}^{n}{}' \frac{1}{2} A_{4lq+4ml+k} (w^{k-4ml} + w^{-k-4ml}) + \Theta \tag{7.3.10}$$

where

$$\Theta = \left[\sum_{k=0}^{n-1}{}' \sum_{j=l+1}^{\infty} (A_{4jq-k} + A_{4jq+k})(\frac{w^k + w^{-k}}{2}) - \right.$$
$$\left. \sum_{k=0}^{n}{}' \sum_{j=l+1}^{\infty} (A_{4jq+4mj-k} + A_{4jq+4mj+k})(\frac{w^k + w^{-k}}{2}) w^{-4ml}\right].$$

Note that for $R > 1$,

$$\Theta = \mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{(4m(l+1)-1)n}}\right). \tag{7.3.11}$$

Let $R \geq \rho$, thus from (7.3.10) and (7.3.11) we have

$$\begin{aligned}
h(z) &= \frac{1}{2}\left(\sum_{k=0}^{n-1}{}' - \sum_{k=-4ml}^{n-4ml}{}'\right) A_{4lq-k} w^k + \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{4lq}} + \frac{R^n}{(\rho-\epsilon)^{4lq+n}} + \right. \\
&\quad \left. + \frac{1}{(\rho-\epsilon)^{4lq}} + \frac{R^n}{(\rho-\epsilon)^{4lq+n}}\right) + \mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{(4m(l+1)-1)n}}\right) \\
&= \frac{1}{2}\left(\sum_{k=n+1-4ml}^{n-1} - \sum_{k=-4ml}^{-1}{}'\right) A_{4lq-k} w^k + \\
&\quad + \mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{4lmn+n}} + \frac{R^n}{(\rho-\epsilon)^{(4m(l+1)-1)n}}\right) \\
&= \frac{1}{2}\sum_{k=0}^{4ml-2} A_{4lq-k-(n+1-4ml)} w^{k+n+1-4ml} + \mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{4lmn+n}}\right).
\end{aligned} \tag{7.3.12}$$

Note that for $R \geq \rho$, by choosing ϵ sufficiently small we can find $\eta_1 > 0$ such that

$$\frac{R^n}{(\rho-\epsilon)^{4lmn+n}} < \left(\frac{R}{\rho^{4lm-1}} - \eta_1\right)^n. \tag{7.3.13}$$

Similarly for $1 < R \leq \rho$ from (7.3.10) and (7.3.11) we have

$$\begin{aligned}
h(z) &= \frac{1}{2}\left(\sum_{k=0}^{n-1}{}' - \sum_{k=-4ml}^{n-4ml}{}'\right) A_{4lq-k} w^k + \mathcal{O}\left(\frac{R^{-n}}{(\rho-\epsilon)^{4lq-n}} + \frac{1}{(\rho-\epsilon)^{4lq}} + \right. \\
&\quad \left. + \frac{R^{-n}}{(\rho-\epsilon)^{4lq-n}} + \frac{1}{(\rho-\epsilon)^{4lq}}\right) + \mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{(4m(l+1)-1)n}}\right) \\
&= \frac{1}{2}\left(\sum_{k=n+1-4ml}^{n-1} - \sum_{k=-4ml}^{-1}{}'\right) A_{4lq-k} w^k + \\
&\quad \mathcal{O}\left(\frac{R^{-n}}{(\rho-\epsilon)^{4lmn-n}} + \frac{R^n}{(\rho-\epsilon)^{(4m(l+1)-1)n}}\right) \\
&= \frac{1}{2}\sum_{k=0}^{4ml-2} A_{4lq-k-(n+1-4ml)} w^{k+n+1-4ml} + \mathcal{O}\left(\frac{R^{-n}}{(\rho-\epsilon)^{4lmn-n}}\right).
\end{aligned} \tag{7.3.14}$$

Again for $1 < R \le \rho$, by choosing ϵ sufficiently small we can find $\eta_2 > 0$ such that

$$\frac{R^{-n}}{(\rho-\epsilon)^{4lmn-n}} < \left(\frac{R}{\rho^{4lm-1}} - \eta_2\right)^n. \tag{7.3.15}$$

Thus from (7.3.12), (7.3.13), (7.3.14) and (7.3.15) we have,

$$h(z) = \frac{1}{2}\sum_{k=0}^{4ml-2} A_{4lq-k-(n+1-4ml)} w^{k+n+1-4ml} + \mathcal{O}\left(\frac{R}{\rho^{4lm-1}} - \eta\right)^n \tag{7.3.16}$$

where η is a positive number.

If we assume that in (7.3.2) equality does not hold at more than $4ml-2$ points, say, $4ml-1$ points, that is

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\lambda}(z_j : f)|^{1/n} < \frac{|w_j|}{\rho^{4ml-1}}, \qquad j = 1, 2, \ldots, 4ml-1$$

for $z_1, \ldots, z_{4ml-1}$ with images $|w_1|, \ldots, |w_{4ml-1}| > 1$ then we have

$$\overline{\lim_{n\to\infty}} |h(z_j)|^{1/n} < \frac{|w_j|}{\rho^{4ml-1}}, \qquad j = 1, 2, \ldots, 4ml-1$$

and hence from (7.3.16)

$$\sum_{k=0}^{4ml-2} A_{4lq-k-(n+1-4ml)} w_j^{k+n+1-4ml} = \beta_{j,n}, \qquad j = 1, \ldots, 4ml-1$$

where

$$|\beta_{j,n}| < K_1 \left(\frac{|w_j|}{\rho^{4ml-1}} - \eta_1\right)^n$$

for some $\eta_1 > 0, K_1 \ge 1, 1 \le j \le 4ml-1$ and $n = 1, 2, \ldots$. Thus

$$\sum_{k=0}^{4ml-2} A_{4lq-k-(n+1-4ml)} w_j^k = w_j^{-n+4ml-1} \beta_{j,n}. \qquad j = 1, \ldots, 4ml-1$$

Solving this system of equations for $A_{4lq-k-(n+1-4ml)}$, $k = 0, 1, \ldots, 4ml-2$, we have

$$A_{4lq-k-(n+1-4ml)} = \sum_{j=1}^{4ml-1} c_j^{(k)} w_j^{-n-4ml-1} \beta_{j,n}$$

where $c_j^{(k)}$ are constants independent of n, which gives

$$\overline{\lim_{n\to\infty}} |A_{4lq-k-(n+1-4ml)}|^{\frac{1}{4lq-k-(n+1-4ml)}}$$

$$\le \overline{\lim_{n\to\infty}} \left(K_2 |w_j|^{-n} \left(\frac{|w_j|}{\rho^{4ml-1}} - \eta_1\right)^n\right)^{\frac{1}{4lq-k-(n+1-4ml)}}$$

$$\le \overline{\lim_{n\to\infty}} \left(K_2 \left(\frac{1}{\rho^{4ml-1}} - \frac{\eta_1}{max|w_j|}\right)^n\right)^{\frac{1}{4lq-k-(n+1-4ml)}}$$

$$< \frac{1}{\rho}$$

which is a contradiction to $f \in A(C_\rho)$. Hence our assumption that equality in (7.3.2) does not hold at more than $4ml - 2$ points was wrong. Thus

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\frac{1}{2}}(z:f)|^{1/n} = \frac{R}{\rho^{4ml-1}}, \qquad z \in C_R, R > 1$$

for all but at most $4lm - 2$ points outside $[-1, 1]$.

Remark 7.3.1 Note that for $\lambda \neq \frac{1}{2}$

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\lambda}(z:f)|^{1/n} = \frac{R}{\rho^{2ml-1}}, \qquad z \in C_R, R > 1$$

for all but at most $2lm - 2$ points outside $[-1, 1]$. That is

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\lambda}(z:f)|^{1/n} < \frac{R}{\rho^{2ml-1}}, \qquad z \in C_R, R > 1$$

for at most $2lm - 2$ points outside $[-1, 1]$. Thus for $R > \rho^{2ml-1}$ and $\lambda \neq \frac{1}{2}$

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\lambda}(z:f)|^{1/n} < B, \qquad z \in C_R, R > \rho^{2ml-1}$$

for at most $2lm - 2$ points, where $B > 1$. Similarly for $\lambda = \frac{1}{2}$ from Theorem 7.3.1 we have for $R > \rho^{4ml-1}$

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\frac{1}{2}}(z:f)|^{1/n} < B, \qquad z \in C_R, R > \rho^{4ml-1}$$

for at most $4lm - 2$ points, where $B > 1$. Thus, we can say that

Theorem 7.3.2 *If $f \in A(C_\rho)$ and $l \geq 1$, then for $\lambda \neq \frac{1}{2}$ and $ml > 1$ the sequence $\{\Gamma_{n-1,l,q,\lambda}(z:f)\}_{n=1}^{\infty}$ can bounded at most at $2ml - 2$ points outside $C_R, R = \rho^{2ml-1}$ and for $\lambda = \frac{1}{2}$ and $ml \geq 1$ the sequence $\{\Gamma_{n-1,l,q,\frac{1}{2}}(z:f)\}_{n=1}^{\infty}$ can bounded at most at $4ml - 2$ points outside $C_R, R = \rho^{4ml-1}$.*

Now we show the sharpness of Theorem 7.3.1 in the sense that

Theorem 7.3.3 *Let $\rho > 1$, $l \geq 1$.*

(i) For $\lambda \neq \frac{1}{2}$, if $ml > 1$ and $z_1, \ldots, z_{2lm-2}$ are arbitrary $2lm-2$ points lying outside $[-1, 1]$. Let $w_1, \ldots, w_{2ml-2}$ be their images in the w-plane defined by the mapping

$$z = \frac{w + w^{-1}}{2}$$

then there is a function $f \in A(C_\rho)$ with

$$\overline{\lim_{n\to\infty}}|\Gamma_{n-1,l,q,\lambda}(z_j;f)|^{1/n} < \frac{|w_j|}{\rho^{2lm-1}}, \qquad for \lambda \neq \frac{1}{2} \text{ and } j = 1,2,\ldots,2lm-2.$$

(ii) *For $\lambda = \frac{1}{2}$ and $ml \geq> 1$, if $z_1, \ldots, z_{4lm-2}$ are arbitrary $4lm-2$ points lying outside $[-1,1]$. Let $w_1, \ldots, w_{4ml-2}$ be their images in the w-plane defined by the mapping*

$$z = \frac{w + w^{-1}}{2}$$

then there is a function $f \in A(C_\rho)$ with

$$\overline{\lim_{n\to\infty}}|\Gamma_{n-1,l,q,\lambda}(z_j;f)|^{1/n} < \frac{|w_j|}{\rho^{4lm-1}}, \qquad for \lambda = \frac{1}{2} \text{ and } j = 1,2,\ldots,4lm-2.$$

Proof For (i) consider the system of equations

$$\sum_{k=0}^{2lm-2} B_{2lq-k-n} w_j^k = 0, \qquad j = 1,2,\ldots.2lm-2 \tag{7.3.17}$$

where $B_{2lq-k-n}$ are the unknowns and $n > 0$. Also (7.3.17) can be written as

$$\sum_{k=1}^{2lm-2} B_{(2lm-1)n-k+2lc} w_j^k = -B_{(2lm-1)n+2lc}. \qquad j = 1,2,\ldots,2lm-2$$

Solving this for $B_{(2lm-1)n-k+2lc}$, $k = 1,\ldots 2lm-2$ we obtain

$$B_{(2lm-1)n-k+2lc} = c_k B_{(2lm-1)n+2lc}, \qquad k = 1,\ldots,2lm-2, n > 0 \tag{7.3.18}$$

where c_k are constants independent of n. Let $c_0 = 1$ and

$$f(z) = \sum_{k=0}^{\infty}{}' A_k T_k(z)$$

where

$$A_{(2lm-1)n-k+2lc} = \frac{c_k}{\rho^{(2lm-1)n+2lc}}, \qquad k = 0,1,\ldots,2lm-2.$$

Then $f \in A(C_\rho)$. Since $c_0 = 1$ thus A_k satisfy (7.3.18) and hence (7.3 17) That is

$$\sum_{k=0}^{2lm-2} A_{2lq-k-n} w_j^k = 0. \qquad j = 1,2,\ldots,2lm-2 \tag{7.3.19}$$

For any $n > 0$ let r and s be determined by

$$2lmn - s = (2lm-1)r. \qquad 0 \leq s \leq 2lm-2$$

Thus from (7.3.19) for $n > 0$, we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1}{}' A_{2lq-k} w_j^k &= \sum_{k=0}^{s-1}{}' A_{2lq-k} w_j^k + \sum_{k=s}^{n-1} A_{2lq-k} w_j^k \\
&= \sum_{k=0}^{s-1}{}' A_{2lmn-k+2lc} w_j^k + \sum_{k=s}^{n-1} A_{2lmn-k+2lc} w_j^k \\
&= \sum_{k=0}^{s-1}{}' A_{2lmn-k+2lc} w_j^k + \\
&\quad \sum_{p=0}^{r-n-1} w_j^{4lmn-(2lm-1)(r-p)} \sum_{k=0}^{2lm-2} A_{(2lm-1)(r-p)-k+2lc} w_j^k \\
&= \sum_{k=0}^{s-1}{}' A_{2lmn-k+2lc} w_j^k + 0 \qquad \text{(from (7.3.19))} \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{2lmn}}\right). \qquad (7.3.20)
\end{aligned}$$

Thus from (7.2.6), (7.3.6) and (7.3.20) for $R \geq \rho$ we have

$$\begin{aligned}
\Gamma_{n-1,l,q,\lambda}(z_j; f) &= \sum_{k=0}^{n-1}{}' \sum_{i=l}^{\infty} \lambda_i (A_{2iq-k} + A_{2iq-k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right) \\
&= \sum_{k=0}^{n-1}{}' \lambda_l (A_{2lq-k} + A_{2lq+k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right) + \\
&\quad \sum_{k=0}^{n-1}{}' \sum_{i=l+1}^{\infty} \lambda_i (A_{2iq-k} + A_{2iq+k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right) \\
&= \sum_{k=0}^{n-1}{}' \frac{1}{2} \lambda_l A_{2lq-k} w_j^k + \sum_{k=0}^{n-1} \frac{1}{2} \lambda_l A_{2lq-k} w_j^{-k} + \\
&\quad \sum_{k=0}^{n-1}{}' \frac{1}{2} \lambda_l A_{2lq+k} (w_j^k + w_j^{-k}) \\
&\quad + \sum_{k=0}^{n-1}{}' \sum_{i=l+1}^{\infty} \lambda_i (A_{2iq-k} + A_{2iq+k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right) \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{2lmn}} + \frac{1}{(\rho-\epsilon)^{2lmn}} + \frac{R^n}{(\rho-\epsilon)^{(2lm+1)n}} + \right. \\
&\quad \left. \frac{R^n}{(\rho-\epsilon)^{(2(l+1)m-1)n}}\right) \\
&= \mathcal{O}\left(\frac{R^n}{(\rho-\epsilon)^{(2lm+1)n}}\right) \\
&= \mathcal{O}\left(\frac{R}{\rho^{2lm-1}} - \eta\right)^n \qquad (7.3.21)
\end{aligned}$$

where η is some positive number.

Similarly for $R \leq \rho$ from (7.2.6), (7.3.8) and (7.3.20) we have

$$\Gamma_{n-1,l,q,\lambda}(z_j; f) = \sum_{k=0}^{n-1}{}' \sum_{i=l}^{\infty} \lambda_i (A_{2iq-k} + A_{2iq+k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1}{}' \lambda_l (A_{2lq-k} + A_{2lq+k}) \left(\frac{w_j^k + w_j^{-k}}{2} \right) + \\
&\quad \sum_{k=0}^{n-1} \sum_{i=l+1}^{\infty} \lambda_i (A_{2iq-k} + A_{2iq+k}) \left(\frac{w_j^k + w_j^{-k}}{2} \right) \\
&= \sum_{k=0}^{n-1}{}' \frac{1}{2} \lambda_l A_{2lq-k} w_j^k + \sum_{k=0}^{n-1} \frac{1}{2} \lambda_l A_{2lq-k} w_j^{-k} + \\
&\quad \sum_{k=0}^{n-1}{}' \frac{1}{2} \lambda_l A_{2lq+k} (w_j^k + w_j^{-k}) \\
&\quad + \sum_{k=0}^{n-1}{}' \sum_{i=l+1}^{\infty} \lambda_i (A_{2iq-k} + A_{2iq+k}) \left(\frac{w_j^k + w_j^{-k}}{2} \right) \\
&= \mathcal{O} \left(\frac{1}{(\rho-\epsilon)^{2lmn}} + \frac{R^{-n}}{(\rho-\epsilon)^{2lmn-n}} + \frac{1}{(\rho-\epsilon)^{2lmn}} + \right. \\
&\quad \left. + \frac{R^n}{(\rho-\epsilon)^{(2(l+1)m-1)n}} \right) \\
&= \mathcal{O} \left(\frac{R^{-n}}{(\rho-\epsilon)^{(2lm-1)n}} \right) \\
&= \mathcal{O} \left(\frac{R}{\rho^{2lm-1}} - \eta \right)^n \qquad (7.3.22)
\end{aligned}
$$

where η is some positive number.

Hence from (7.3.21) and (7 3.22)

$$\overline{\lim_{n\to\infty}} |\Gamma_{n-1,l,q,\lambda}(z_j; f)|^{1/n} < \frac{|w_j|}{\rho^{2lm-1}}, \qquad for \lambda \neq \frac{1}{2} \ and \ j = 1, 2, \ldots . 2lm-2.$$

For (ii) consider the system of equations

$$\sum_{k=0}^{4lm-2} B_{4lq-k-n} w_j^k = 0, \qquad j = 1, 2, \ldots, 4lm-2 \qquad (7.3.23)$$

where $B_{4lq-k-n}$ are the unknowns and $n > 0$. Also (7.3.23) can be written as

$$\sum_{k=1}^{4lm-2} B_{(4lm-1)n-k+4lc} w_j^k = -B_{(4lm-1)n+4lc}. \qquad j = 1, 2, \ldots . 4lm-2$$

Solving this for $B_{(4lm-1)n-k+4lc}$, $k = 1, \ldots 4lm-2$ we obtain

$$B_{(4lm-1)n-k+4lc} = c_k B_{(4lm-1)n+4lc}, \qquad k = 1, \ldots, 4lm-2, n > 0 \qquad (7.3.24)$$

where c_k are constants independent of n. Let $c_0 = 1$ and

$$f(z) = \sum_{k=0}^{\infty}{}' A_k T_k(z),$$

where

$$A_{(4lm-1)n-k+4lc} = \frac{c_k}{\rho^{(4lm-1)n+4lc}}, \qquad k = 0, 1, \dots, 4lm-2.$$

Then $f \in A(C_\rho)$. Since $c_0 = 1$ thus A_k satisfy (7.3.24) and hence (7.3.23). That is

$$\sum_{k=0}^{4lm-2} A_{4lq-k-n} w_j^k = 0. \qquad j = 1, 2, \dots, 4lm-2 \tag{7.3.25}$$

For any $n > 0$ let r and s be determined by

$$4lmn - s = (4lm-1)r. \qquad 0 \le s \le 4lm-2$$

Thus from (7.3.25) for $n > 0$, we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1}{}' A_{4lq-k} w_j^k &= \sum_{k=0}^{s-1} A_{4lq-k} w_j^k + \sum_{k=s}^{n-1} A_{4lq-k} w_j^k \\
&= \sum_{k=0}^{s-1} A_{4lmn-k+4lc} w_j^k + \sum_{k=s}^{n-1} A_{4lmn-k+4lc} w_j^k \\
&= \sum_{k=0}^{s-1} A_{4lmn-k+4lc} w_j^k + \\
&\quad \sum_{p=0}^{r-n-1} w_j^{4lmn-(4lm-1)(r-p)} \sum_{k=0}^{4lm-2} A_{(4lm-1)(r-p)-k+4lc} w_j^k \\
&= \sum_{k=0}^{s-1} A_{4lmn-k+4lc} w_j^k + 0 \qquad \text{(from (7.3.25))} \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{4lmn}}\right).
\end{aligned} \tag{7.3.26}$$

Thus from (7.2.7), (7.3.13) and (7.3.26) for $R \ge \rho$ we have

$$\begin{aligned}
\Gamma_{n-1,l,q,\lambda}(z_j; f) &= \sum_{k=0}^{n-1}{}' \sum_{i=l}^{\infty} (A_{4iq-k} + A_{4iq+k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right) \\
&= \sum_{k=0}^{n-1}{}' (A_{4lq-k} + A_{4lq-k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right) + \\
&\quad \sum_{k=0}^{n-1}{}' \sum_{i=l+1}^{\infty} (A_{4iq-k} + A_{4iq+k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right) \\
&= \sum_{k=0}^{n-1}{}' \frac{1}{2} A_{4lq-k} w_j^k + \sum_{k=0}^{n-1}{}' \frac{1}{2} A_{4lq-k} w_j^{-k} + \\
&\quad \sum_{k=0}^{n-1}{}' \frac{1}{2} A_{4lq+k} (w_j^k + w_j^{-k}) \\
&\quad + \sum_{k=0}^{n-1}{}' \sum_{i=l+1}^{\infty} (A_{4iq-k} + A_{4iq+k}) \left(\frac{w_j^k + w_j^{-k}}{2}\right) \\
&= \mathcal{O}\left(\frac{1}{(\rho-\epsilon)^{4lmn}} + \frac{1}{(\rho-\epsilon)^{4lmn}} + \frac{R^n}{(\rho-\epsilon)^{(4lm+1)n}} + \right.
\end{aligned}$$

then from Theorem 7.2.2 we have

$$B_{l,m,\lambda}(z;f) \le \frac{R}{\rho^{2ml-1}}, \qquad z \in C_R \ and \lambda \neq \frac{1}{2}$$

and

$$B_{l,m,\lambda}(z;f) \le \frac{R}{\rho^{4ml-1}}, \qquad z \in C_R \ and \lambda = \frac{1}{2}.$$

If for $\lambda \neq \frac{1}{2}$ we set

$$\delta_{l,m,\rho,\lambda}(f) = \{z | B_{l,m,\lambda}(z;f) < \frac{R}{\rho^{2ml-1}}\}, \qquad f \in A(C_\rho), \ \ \rho > 1$$

and for $\lambda = \frac{1}{2}$ we set

$$\delta_{l,m,\rho,\frac{1}{2}}(f) = \{z | B_{l,m,\frac{1}{2}}(z;f) < \frac{R}{\rho^{4ml-1}}\}, \qquad f \in A(C_\rho), \ \ \rho > 1$$

then from Theorem 7.3.1 for $\lambda \neq \frac{1}{2}$

$$|\delta_{l,m,\rho,\lambda}(f) \cap \{z | z \in C_R, R > 1\}| \le 2ml - 2$$

and for $\lambda = \frac{1}{2}$

$$|\delta_{l,m,\rho,\frac{1}{2}}(f) \cap \{z | z \in C_R, R > 1\}| \le 4ml - 2$$

where $|S|$ denotes the cardinality of set S. Thus if we designate a set Z of points as "$(l, m, \rho, \lambda) - distinguished$" if there is an $f \in A(C_\rho)$ such that $Z \in \delta_{l,m,\rho,\lambda}(f)$. Theorem 7.3.3 can be stated as

Theorem 7.3.4 *For $\lambda \neq \frac{1}{2}$ and $ml > 1$ any set of $2ml - 2$ points outside $[-1, 1]$ is $(l, m, \rho, \lambda)-$ distinguished and for $\lambda = \frac{1}{2}$ and $ml \ge 1$ any set of $4ml - 2$ points outside $[-1, 1]$ is $(l, m, \rho, \frac{1}{2})-$ distinguished.*

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